Distribution of the logarithms of currents in percolating resistor networks. I. Theory

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The distribution of currents, i_b, in the bonds b of a randomly diluted resistor network at the percolation threshold is investigated through a study of the moments of the distribution $\hat{P}(i^2)$ and the moments of the distribution $P(y)$, where $y=-\ln i_b^2$. For $q>q_c$ the qth moment of $\hat{P}(i^2)$, M_q (i.e., the average of i^{2q}), scales as a power law of the system size L, with a multifractal (noise) exponent $\tilde{\psi}(q)-\tilde{\psi}(0)$. Numerical data indicate that q_c is negative, but becomes small for large L. Assuming that all derivatives $\tilde{\psi}(q)$ exist at $q=0^+$, we show that for positive integer k the kth moment, μ_k, of $P(y)$ is given by

$$\mu_k=(\alpha_0 \ln L)^k\{1+[kC_1+\tfrac{1}{2}k(k-1)D_1](\ln L)^{-1}+O[(\ln L)^{-2}]\} ,$$

where α_0 and D_1 (but not C_1) are *universal* constants obtained from $\tilde{\psi}(q)$. A second independent argument, requiring an assumed analyticity property of the asymptotic multifractal function, $f(\alpha)$, leads to the same equation for *all* k. This latter argument allows us to include finite-size corrections to $f(\alpha)$, which are of order $(\ln L)^{-1}$. These corrections must be taken into account in interpreting numerical studies of $P(y)$. We note that data for $P(-\ln i^2)$ seem to show power-law behavior as a function of i^2 for small i. Values of the exponents are directly related to the values of q_c, and the numerical data in two dimensions indicate it to be small (but probably nonzero). We suggest, in view of the nature of the finite-size corrections in the expression for μ_k, that the asymptotic regime may not have been reached in the numerical work. For $d=6$ we find that $M_q(L)\sim(\ln L)^{\theta(q)}$, where $\theta(q)\rightarrow\infty$ for $q\rightarrow q_c=-\frac{1}{2}$.

I. INTRODUCTION

Recently there has been increasing interest in the multifractal description of probability distributions on fractal structures. Originally proposed to treat nonuniform turbulence,[1] this formulation provides a mathematical framework within which it is possible to discuss systematically families of fractal measures which may be used to characterize a fractal set. For example, fractal measures have been introduced to describe the nonuniform growth probabilities in diffusion limited aggregation (DLA),[2-4] strange attractors in dynamical systems,[5,6] and localized wave functions of particles in a random potential,[7] to name but a few. One of the first and most systematically studied cases concerns the distribution of currents in bonds of a randomly diluted resistor network at the percolation threshold,[8-20] which we will refer to as a percolating resistor network.

Roughly speaking, and as we will see in more detail later, a multifractal distribution is one which displays power-law scaling as a function of the system size, L, but with continuously variable exponents, $f(\alpha)$, associated with different regions (α) of the distribution. One central question with regard to any of these systems is the degree to which the multifractal formalism provides a complete description of the entire distribution function for the stochastic variable. As we will discuss in more detail later, while the multifractal formalism does describe most of the important features of the distribution function, some of the finer details are not naturally contained in this framework. This situation is perhaps analogous to attempting to describe the Griffiths singularity[21] in dilute Ising systems within the renormalization-group framework. In both the random resistor network and the dilute Ising model, one has a dominant power-law behavior which is naturally treated in a multifractal or renormalization-group approach. This approach works best when properties at large length scales can be obtained recursively from those at small length scales. However, rigorous arguments[12,21] also indicate the presence of tails in the distribution function which occur with exponentially small probability and which are thus not easily accessible to such recursive formalisms. If the stochastic variable is denoted by x (in the following $x=i_b^2$, where i_b is the current in the bond b of the percolating resistor network), then a multifractal distribution of x implies that the qth moment of x, denoted $M_q(L)$, varies as a power of L with a q-dependent exponent. As we discuss below, large percolating networks have currents of order $i_{\min}^2\propto\exp(-KL^{\rho})$, which result in an exponential growth of $M_q(L)$ with L, for $q<q_{c_1}\leq 0$. Strong arguments,[12] presented below, indicate that the threshold q_{c_1} is nonzero and negative. However, other small currents,

which decay with L faster than a power law but slower than an exponential, may generate deviations from the power-law dependence of $M_q(L)$, or from multifractality, below a higher threshold q_c. As we discuss below, it seems most likely that q_c is strictly negative ($q_c<0$, not $q_c=0$).

A second question, which recent work[10] has suggested, is whether or not the rather complicated multifractal scenario could be completely obviated by considering the distribution for $\ln x$. In particular, one might ask whether this distribution would be a function of a reduced variable of the form $\ln x/(\ln L)^{\hat{\phi}}$, where $\hat{\phi}$ is a crossover exponent. *A priori* it is not clear that one can generally hope to reduce all the information in an entire multifractal function to a *unifractal* one, i.e., to one with a single scaling exponent, $\hat{\phi}$. In view of these questions, this paper is devoted to a study of the distribution function for the logarithms of the currents in percolating resistor networks. A convenient way to access the distribution of $\ln x$ is to consider the kth moment of $\ln x$, μ_k. Our arguments show that μ_k scales as $(\ln L)^k$, i.e., $\hat{\phi}=1$. Thus, the dominant scaling behavior of the distribution of $\ln x$ is indeed much simpler than that of x. However, as we shall see, multifractality in the distribution of i_b^2 is reflected in finite-size corrections of order $(\ln L)^{-1}$ to the cumulants of the μ_k's, denoted μ_k^c.

We thus carry out a detailed analysis of μ_k, including corrections of relative order $k/\ln L$ and $k(k-1)/\ln L$. In Secs. III and IV, we give two independent arguments that

$$\mu_k=(\alpha_0\ln L)^k\{1+[kC_1+\tfrac{1}{2}k(k-1)D_1](\ln L)^{-1} +O[(\ln L)^{-2}]\}\ , \tag{1.1}$$

where the constants will be discussed in more detail below. We see that the asymptotic behavior is only reached when $\ln L>>|k|$, $|k(k-1)|$. The first derivation, which is valid for positive integer k, invokes derivatives of $M_q(L)$ with respect to q. This derivation involves assuming the existence for $q\to 0^+$ of derivatives of $\tilde{\psi}(q)$, the exponent which describes the scaling with L of $M_q(L)$. Although this assumption can be shown to fail for some special models,[14] we believe that for the random resistor network on the percolating cluster, such an assumption of regularity holds, as we will discuss in Sec. III. The second derivation, presented in Sec. IV, depends on an assumption that the multifractal function $f(\alpha)$ [defined in Eq. (2.9), below] can be expanded about its maximum at $\alpha=\alpha_0$ in powers of $(\alpha-\alpha_0)$. Since we believe the assumption of regularity of $\tilde{\psi}(q)$ for $q\to 0^+$ is true, and therefore that Eq. (1.1) is true, we suggest that the assumption that $f(\alpha)$ is analytic at $\alpha=\alpha_0$ is probably also true. Although some numerical work[16] would suggest that q_c is very small, we use the above reasoning to argue in Sec. IV C that $\tilde{\psi}(q)$ is an analytic function of q for $q>q_c$, where q_c is strictly negative.

Our second derivation of Eq. (1.1) is based on a new extension of the multifractal analysis, to include finite-size corrections to $f(\alpha)$. We show that these corrections can be expanded systematically in powers of $(\ln L)^{-1}$, give explicit expressions for these corrections, and emphasize their importance for measurements over limited ranges of sizes. This formulation thus provides a quantitative theoretical explanation for the slow convergence of $f(\alpha)$ with increasing system size observed in numerical work.[11]

Briefly, this paper is organized as follows. In Sec. II, we review the multifractal formalism as applied to the percolating resistor network. In particular, we note the existence of non-power-law scaling which is outside the "standard" multifractal formalism. In Sec. III, we obtain Eq. (1.1) from the scaling of the cumulant moments of $\ln x$, which, in turn, are obtained in terms of derivatives at $q=0^+$ of $M_q(L)$. In Sec. IV, we present some new extensions of the multifractal formalism to include finite-size corrections. This formalism is applied to obtain a second independent derivation of Eq. (1.1). In Sec. V, we give an extensive discussion of the existence of q_c. We discuss its relation to the probability distribution for the currents and how it can be extracted from numerical work which we review in some detail. Our conclusions are summarized in Sec. VI. In the Appendix we obtain and discuss the exact solution for this distribution function for the Mandelbrot-Given[22] fractal (MGF), which corroborates Eq. (1.1).

II. REVIEW OF MULTIFRACTALITY

We start with a brief review of multifractality as it applies to the percolating resistor network. We consider a network of nodes forming a hypercubic lattice in d spatial dimensions. Initially nearest-neighboring nodes (whose separation, a, is taken to be unity) are connected by unit conductances. After random dilution each conductance randomly assumes the values 1 and 0 with respective probabilities p and $1-p$. If a unit current[23] is inserted into the network at node $\mathbf{x}$ and removed at node $\mathbf{x}'$, a current, $i_b(\mathbf{x},\mathbf{x}')$, appears in the bond b. We then define[13]

$$M_q(\mathbf{x},\mathbf{x}')=\left[\sum_b [i_b(\mathbf{x},\mathbf{x}')]^{2q}\right]_{\rm av}\Bigg/\left[\sum_b 1\right]_{\rm av}\ , \tag{2.1}$$

where the sums run over all bonds b with nonzero currents i_b and $[\cdots]_{\rm av}$ indicates an average over all configurations of conductances. Fractal, or power-law, behavior occurs for $1\ll|\mathbf{x}-\mathbf{x}'|\ll\xi_p$, where ξ_p is the percolation correlation length, which measures the size of typical structures in the randomly diluted network.[24] In this paper we confine our analysis to the percolation threshold, $p=p_c$, where $\xi_p=\infty$, and therefore we are always in the fractal regime. Paper II in this series[25] will discuss concentrations $p<p_c$. In the fractal regime, multifractal behavior implies that

$$M_q(\mathbf{x},\mathbf{x}')=\tilde{A}_q|\mathbf{x}-\mathbf{x}'|^{\tilde{\psi}(q)-\tilde{\psi}(0)} \tag{2.2a}$$

in the asymptotic limit $|\mathbf{x}-\mathbf{x}'|\to\infty$. In Eq. (2.2a), $\tilde{A}_q$ is a nonuniversal amplitude and $\tilde{\psi}(q)$ are the *multifractal* exponents,[8-13] which do not depend on $|\mathbf{x}-\mathbf{x}'|$. In particular, $\tilde{\psi}(0)$ is the fractal dimension of the backbone, D_B. [In Refs. 8, 9, 14, 15, and 18 the notation $\tilde{\psi}(q)=-x_q$ is used.] One may also consider the value of M_q for a system of linear dimension L when the nodes $\mathbf{x}$ and $\mathbf{x}'$ are separated by a distance of order L. Finite-size scaling indicates that this quantity is given by

$$M_q(L) \approx A_q L^{\tilde{\psi}(q)-\tilde{\psi}(0)} , \qquad (2.2b)$$

where A_q is another nonuniversal amplitude. For the other problems mentioned above, one similarly constructs moments of the appropriate distribution functions, e.g., of the growth probabilities for DLA.[2-4]

Note that Eq. (2.2b) is expected to hold asymptotically for large L. As usual in critical phenomena, one expects corrections with smaller powers of L. However, these corrections will be ignored, since power-law corrections have a negligible effect on the results obtained in this paper. In particular, except at special values of d (see Sec. V B), there are no logarithmic corrections to Eq. (2.2b).

For positive values of q all approaches,[8-13] numerical or analytic, are in qualitative agreement with Eq. (2.2). However, for negative q the situation is much less clear because in this case M_q is dominated by extremely small currents. In fact, Blumenfeld *et al.*[12] (BMAH) showed that for q sufficiently negative, Eq. (2.2b) ceases to hold and is replaced by

$$\ln M_q(L) \sim -\ln i^2_{\min} \sim L^{\rho} , \qquad (2.3a)$$

where $i_{\min}$ is the smallest current in the network and ρ is an exponent whose value was roughly estimated. Presumably there exists a negative critical value of q, which we denote q_{c1}, such that Eq. (2.3a) holds for $q < q_{c1}$. *A priori,* it is not clear whether Eq. (2.2b) holds for all $q > q_{c1}$. There may exist[26] an interval $q_{c1} < q < q_c$, in which $M_q(L)$ grows faster than a power law, but slower than exponentially, e.g.,

$$\ln M_q(L) \sim (\ln L)^{\omega} , \qquad (2.3b)$$

with[26] $\omega > 1$. In the case of DLA, there is still some controversy[4,27-32] over the analogous situation. Initially[27] it was suggested that for DLA q_{c_1} and possibly also q_c were equal to $-\infty$, but it was also suggested that $q_c = 0$.[4,28] In fact, more complicated scenarios with the behavior given in Eq. (2.3b) have been proposed.[29] In some scenarios, one even found that $q_c(L)$ may approach zero from above.[31] Very recent large simulations[32] seem to confirm the behavior in Eq. (2.3a). As discussed in Sec. IV, our work indicates that for percolation clusters q_c is strictly negative (i.e., bounded away from zero). This result does not depend on the explicit asymptotic form of $M_q(L)$ for $q < q_c$.

The moments $M_q(L)$ contain information on the underlying *asymptotic distribution* function which we would like to access. In fact, if one writes

$$M_q(L) = \int_0^1 d(i^2) \hat{P}(i^2, L) i^{2q} , \qquad (2.4)$$

then the distribution function, $\hat{P}(i^2,L)$, can be obtained[14,15] by inverting this relation, as in the famous "moment" problem of mathematical analysis. Although this is possible, in principle,[14,15] it is difficult[18] precisely because of the difficulty in treating the extremely small currents which only occur with small probability. Such small currents are often neglected in many treatments because they only have a minute effect on the positive q moments, which are usually the object of interest. In Sec. V D, below, we show that these currents lead to a power-law behavior of $\hat{P}(i^2,L)$ for small i^2.

Early attempts to estimate $\hat{P}(i^2,L)$ used hierarchical structures to imitate the spanning percolation cluster. One example is the MGF shown in Fig. 1. In a simpler version, one has two equal bonds in parallel in the central section of each iteration.[10] After N iterations of the simpler version, the linear size of the structure becomes $L = 3^N$, and the currents assume the values $i_k = 2^{-k}$, with probabilities

$$\hat{P}(i_k^2, L) = 2^{-N} \begin{bmatrix} N \\ k \end{bmatrix} . \qquad (2.5)$$

A similar, although somewhat more complicated, expression is obtained for the MGF in the Appendix. For such distributions, the multifractal behavior of Eq. (2.2b) holds for *all* q. Hierarchical structures[26,33] do not faithfully reproduce the anomalous behavior of negative moments as they are observed on percolation clusters. Since $k = -\ln i_k / \ln 2$, Eq. (2.5) represents a *log-binomial distribution,* which behaves like a Gaussian distribution near its maximum, i_m. This fact led the authors of Ref. 10 to propose the *Gaussian approximation*

$$\hat{P}(i^2, L) \sim \exp\left[-\frac{(\ln i^2 - \ln i_m^2)^2}{2\sigma^2} \right] , \qquad (2.6)$$

with $\ln i_m^2 \sim N \sim \ln L$ and $\sigma \sim N \sim \ln L$. However, this approximation fails badly away from the maximum, and therefore is not suitable for calculations of the moments $M_q(L)$. In fact, substitution of Eq. (2.6) into Eq. (2.4) yields a quadratic dependence of $\tilde{\psi}(q)$ on q, instead of approaching a known finite limit as $q \to \infty$.[12] As we show in the Appendix, Eq. (2.6) fails for the MGF. However, a correct analytic analysis of a distribution similar to Eq. (2.5) reproduces all the predictions of the present paper, including our main result, Eq. (1.1). That analysis is very useful, as it also shows explicitly the limits of validity of the finite-size corrections. Although the Gaussian approximation fails, Eq. (2.5) suggests that $\hat{P}(i^2,L)$ may depend on i^2 via a simple function of ln (i^2). One is thus led to study the distribution function of the logarithms of the currents, $y = |\ln i^2|$,

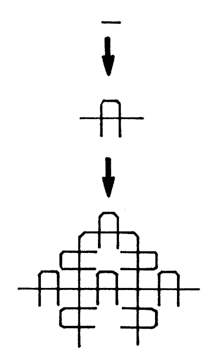

FIG. 1. Three iterations of the Mandelbrot-Given fractal curve. At each iteration, a bond is replaced by eight new bonds and the length scale is changed by a factor of 3.

$$P(y,L)=\hat{P}(i^2,L)|d(i^2)/dy| \; . \tag{2.7}$$

As noted by Fourcade and Tremblay,[14] Eq. (2.4) indicates that $M_q(L)$ is the Laplace transform of $P(y,L)$:

$$M_q(L)=\int_0^{y_{\max}} dy\, P(y,L)e^{-qy} \; . \tag{2.8}$$

For finite L, $y_{\max}=-\ln i^2_{\min}$ is finite. As $L\to\infty$, $y_{\max}\to\infty$. Of course, we are interested in the behavior in the asymptotic limit $L\to\infty$. In that case the only possible dependence on the limit can occur when q is negative and $y_{\max}$ becomes large. (Again, we believe the type of pathology found in Ref. 14 will not actually occur on percolating clusters.) If such a dependence on cutoff occurs, it implies a breakdown of standard multifractality.[4] Using the inverse Laplace transform, and a saddle-point approximation, Fourcade and Tremblay[14] found that in the limit of infinite L, when one extends the integration to the whole y axis, Eq. (2.2b) is equivalent to

$$\lim_{L\to\infty}\frac{\ln P(y,L)}{\ln L}=f(\alpha) \; , \tag{2.9}$$

where

$$\alpha=\frac{y}{\ln L}=-\frac{\ln i^2}{\ln L} \; . \tag{2.10}$$

Here $f(\alpha)$ is the Legendre-transformed function given by[2,12]

$$f(\alpha)=q\alpha+\tilde{\psi}(q)-\tilde{\psi}(0) \; , \tag{2.11}$$

where q is a function of α determined by

$$\alpha=-\frac{\partial\tilde{\psi}}{\partial q} \; . \tag{2.12}$$

For finite L we will extend the definition of $f(\alpha)$ to be

$$f(\alpha,L)=\ln P(y,L)/\ln L \; . \tag{2.13}$$

In Sec. IV we will show that for large L, $f(\alpha,L)-f(\alpha)$ has an expansion in powers of $(\ln L)^{-1}$. If Eq. (2.9) holds and if $f(y/\ln L)$ has no other L dependence for sufficiently large L, then one has data collapse when $\ln P/\ln L$ is plotted versus $\alpha=y/\ln L$. The fact that $f(\alpha,L)$ has corrections at large L of order $(\ln L)^{-1}$ implies that $\ln P/\ln L$ approaches its asymptotic limit $f(\alpha)$ very slowly, with finite-size corrections of relative order $(\ln L)^{-1}$. Such large finite-size corrections give a quantitative explanation for the slow convergence found, e.g., in Ref. 11 for $f(\alpha)$ with increasing L.

Although we have mentioned problems other than the distribution of currents on percolating clusters, there are important reasons for restricting our attention to this case. First of all, from renormalization-group ϵ expansions and scaling treatments, we believe the power-law scaling of Eq. (2.2b) is well established for all $q\geq 0^+$. Such an ansatz may not be valid for arbitrary distributions on arbitrary fractal structures. In particular, for percolating clusters we do not expect pathologies which are mathematically allowed, in general, but which seem unlikely on physical grounds. Our arguments are less well founded in DLA, for example, where power-law scaling has not yet been given a firm theoretical foundation.[31] In the present paper we are therefore studying multifractality in a very controlled setting, although it is not easy to formulate this in rigorous mathematical terms. We therefore specifically warn the reader that assumptions which seem plausible in the context of percolating clusters can often be violated in models which have less direct physical relevance. However, we hope that the present detailed analysis will stimulate analogous studies of DLA.

III. MOMENTS AND CUMULANTS OF $\ln i^2$

We now study the distribution of $\ln i^2$ by a consideration of the moments and cumulants of $y=|\ln i^2|$. The major result of this section is to derive Eq. (1.1) under mild assumptions, which we believe are valid for percolating networks, though not for general multifractal systems. In the Appendix we present an explicit analytic example, in which all of these assumptions are clearly justified. To start, we define the moments, μ_k, by

$$\mu_k(\mathbf{x},\mathbf{x}')=\left[\sum_b{}' |\ln i_b^2|^k\right]_{\rm av}\Big/\left[\sum_b 1\right]_{\rm av} \; , \tag{3.1}$$

where the prime on the sum excludes the singly connected bonds (which have $i_b=1$). As before we consider this quantity on a large length scale, L, which we write as $\mu_k(L)$. Comparison with Eq. (2.1) indicates the relation

$$\mu_k(L)=(-1)^k\left.\frac{\partial^k M_q(L)}{\partial q^k}\right|_{q=0} \; . \tag{3.2}$$

Similarly, the corresponding *cumulants* are found via

$$\mu_k^c(L)=(-1)^k\left.\frac{\partial^k}{\partial q^k}\ln M_q(L)\right|_{q=0} \tag{3.3}$$

(M_q and $\ln M_q$ are similar to the partition function and free energy in statistical mechanics). Assuming that Eq. (2.2b) holds for $q\geq 0$, Eq. (3.3) yields

$$\mu_k^c(L)=(-1)^k\left.\frac{\partial^k}{\partial q^k}\ln A_q\right|_{q=0^+}+\tilde{\psi}^{(k)}\ln L \; , \tag{3.4}$$

where $\tilde{\psi}^{(k)}\equiv(-1)^k\partial^k\tilde{\psi}(q)/\partial q^k|_{q=0^+}$. Thus, *all* the cumulants of y are *linear* in $\ln L$, with coefficients which are directly related to derivatives of the universal function $\tilde{\psi}(q)$ at $q=0^+$. This is one of our main new results.

This result is certainly true if $M_q(L)$ is analytic in q for some finite interval around $q=0$, i.e., if $q_c<0$. However, we expect it to hold even if $q_c=0$, since we believe that *for the percolating cluster* all derivatives of $M_q(L)$ exist for $q\to 0^+$. We now briefly discuss the evidence in favor of this belief. First of all, the renormalization-group ϵ expansion does not show any singularity in $\tilde{\psi}(q)$ at $q=0$. (See Sec. V B, below). Secondly, it is clear that as a function of q, a problem at $q\to 0$ can only come from currents which become very small as the system size becomes large. Here "very small" means small compared to a power of L. In addition, of course, such anomalous currents must occur sufficiently frequently to affect $M_q(L)$. Again, small currents whose probability is exponentially small in L can be neglected. On the percolation cluster, we believe that small currents occur exponentially rarely, as in Ref. 12. In this connection, the model introduced in Ref. 14 [for which $\tilde{\psi}(0^+)$ is *not* regular] seems not to reproduce correctly the behavior of small currents on the percolating cluster. In summary, the evidence seems compelling that $\tilde{\psi}(0^+)$ is completely regular and has derivatives of all orders. We also mention that although Eq. (2.2b) had corrections involving smaller powers of L, these are completely negligible compared to the terms we kept in Eq. (3.4).

We now consider some of the implications of Eq. (3.4). Noting the relation between cumulants and averages, and setting $\mu_1^c(L)=\mu_1(L)$, we see that Eq. (3.4) implies that, for large L,

$$\mu_k(L)=\mu_1(L)^k+\frac{k(k-1)}{2}\mu_1(L)^{k-2}\mu_2^c(L)+O[(\ln L)^{k-2}] \tag{3.5a}$$

$$\approx(\alpha_0\ln L)^k\left[1+C_1\frac{k}{\ln L}+D_1\frac{k(k-1)}{2\ln L}+O[(\ln L)^{-2}]\right], \tag{3.5b}$$

where $\alpha_0=\tilde{\psi}^{(1)}=-\partial\tilde{\psi}(q)/\partial q|_{q=0}$, $D_1=\tilde{\psi}^{(2)}/\alpha_0^2$, and $C_1=-dA_q/dq|_{q=0}/\alpha_0$. In the expression for C_1 we used the fact [see Eqs. (2.1) and (2.2b)] that $A_0=1$. Since $\tilde{\psi}(q)$ is a universal function of q, it follows that α_0 and D_1 are universal constants, but C_1 is nonuniversal. In Eq. (3.5b) the correction term involving C_1 comes from the first term on the right-hand side of Eq. (3.4) for $k=1$ and the correction term involving D_1 comes from the term in Eq. (3.5a) involving μ_2^c.

Some comments about Eq. (3.5b) are in order. The conclusion that

$$\mu_k\sim(\ln L)^{\beta(k)} \tag{3.6}$$

with

$$\beta(k)=k \tag{3.7}$$

is a striking one. As we have said, there seems to be no reason to expect that derivatives of $\tilde{\psi}(q)$ as $q\to 0^+$ could diverge. Also $\ln A_q$ seems to display hardly any dependence on q.[15,20] These observations seem to justify Eq. (3.4), upon whose validity Eq. (3.5) relies. Current numerical data are consistent with Eq. (3.5b). For instance, in $d=3$ for $k>-3$, Duering and Bergman[20] found empirically that $\ln\mu_k(L)$ is practically linear in k (see their Fig. 4), consistent with Eq. (3.5b). Simulations in $d=2$ give[34] $\beta(k)/k=1.15\pm0.06$, with systematic deviations which may be due to higher-order finite-size corrections. Also, as we shall see in paper II,[25] series expansions are consistent with Eqs. (3.6) and (3.7), but at present the data is not conclusive on this point.

The form of Eq. (3.5b) suggests further ways by which it might be tested in the future. For example, one consequence of Eq. (3.5) is that

$$\frac{\mu_{k+1}}{\mu_k}=\alpha_0[\ln L+C_1+D_1k+O(1/\ln L)]\ . \tag{3.8}$$

In fact, α_0, C_1, and D_1 can be measured from such ratios. Further, Eq. (3.5) implies the universality of the ratios

$$\frac{\mu_k\mu_l}{\mu_m\mu_n}=1+D_1[k(k-1)+l(l-1)-m(m-1)-n(n-1)](2\ln L)^{-1}+O[(\ln L)^{-2}],\quad k+l=m+n\ . \tag{3.9}$$

In fact, this latter result has been confirmed by series studies.[25] Finally, we may note where information (e.g., the $\tilde{\psi}^{(k)}$'s) concerning multifractality is contained in the μ_k's. In particular, from the cumulant moments μ_k^c we can determine the $\tilde{\psi}^{(k)}$'s by $\tilde{\psi}^{(k)}=\lim_{L\to\infty}[\mu_k^c/\ln L]$.

The distribution of i_b^2 is multifractal if $D_1\neq 0$. Even if $D_1=0$, i.e., if $\tilde{\psi}^{(2)}=0$, one still has multifractality unless $\tilde{\psi}^{(k)}=0$ for all $k\geq 2$. Such a scenario of multifractality is unlikely. Similarly, Eq. (3.4) indicates that for the cumulant moments $\mu_k^c(L)\sim\tilde{\psi}^{(k)}\ln L$. Thus, multifractality is characterized by $\mu_k^c(L)/\ln L$ being nonzero for some $k\geq 2$.

IV. FINITE-SIZE CORRECTIONS TO $f(\alpha,L)$

A. Systematic expansion of $f(\alpha,L)$ in powers of $\ln L$

Here we consider how $f(\alpha,L)$ should be modified to take proper account of finite-size effects. To do this we develop a systematic expansion of $f(\alpha,L)$ in powers of $(\ln L)^{-1}$. These results can be used in various ways. For instance, the first application we consider is to give an alternative derivation of Eq. (3.5) which sheds more light on the problems related to q_c. The second application is to discuss the characteristic behavior of the finite-size corrections, which have actually been observed in numerical studies.

We start by assuming that $\tilde{\psi}(q)$ is analytic at $q=0$, and has a Taylor expansion that converges in a finite interval of q around $q=0$. This is correct for all finite L and also for all $q\geq q_c$, even as $L\to\infty$. Now let us explore the possibility of consistently keeping track of finite-size corrections in $f(\alpha,L)$. To do this we write

$$\frac{\ln M_q(L)}{\ln L}=\tilde{\psi}(q)-\tilde{\psi}(0)+\frac{\ln A_q}{\ln L}\equiv\tau^{(0)}(q)+\tau^{(1)}(q)(\ln L)^{-1}\equiv\tau(q,L)\ , \tag{4.1}$$

and we will develop the multifractal formalism in terms of $\tau(q,L)$ which contains corrections of order $(\ln L)^{-1}$ to the usual asymptotic formalism. We now invert Eq. (2.8) extending the range of integration over an infinite interval, so that

$$P(y,L)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{qy+\ln M_q(L)}dq \tag{4.2a}$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{[q\alpha+\tau^{(0)}(q)]\ln L}e^{\tau^{(1)}(q)}dq\ , \tag{4.2b}$$

where $\alpha\equiv y/\ln L$, as before. Now the saddle-point occurs at[14] $q=q_L^*\equiv q_L(\alpha)$ and is determined not by Eq. (2.12) but rather by

$$\alpha=-\frac{\partial\tau(q,L)}{\partial q}=-\frac{\partial\tilde{\psi}(q)}{\partial q}-\frac{1}{A_q \ln L}\frac{\partial A_q}{\partial q}\,, \tag{4.3}$$

which incorporates finite-size effects. In evaluating the saddle-point contribution in Eq. (4.2b) the simplest procedure is to expand about the $L=\infty$ saddle [at $q=q^*$ determined by Eq. (2.12)] and treat $\tau^{(1)}(q)$ perturbatively. Keeping quadratic fluctuations about the $L=\infty$ saddle (at $q=q^*_\infty \equiv q^*$) in this way, we get

$$P(y,L)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{[q^*\alpha+\tau(q^*,L)+(q-q^*)\alpha+(q-q^*)\partial\tau/\partial q+(1/2)(q-q^*)^2\partial^2\tau/\partial q^{*2}]\ln L}dq \tag{4.4a}$$

$$=e^{[q^*\alpha+\tau(q^*,L)]\ln L}\frac{1}{2\pi i}\int_{-\infty}^{i\infty} e^{[(1/2)(q-q^*)^2\ln L\,\partial^2\tau/\partial q^{*2}+(q-q^*)\partial\tau^{(1)}/\partial q^*]}dq \tag{4.4b}$$

$$=e^{[q^*\alpha+\tau(q^*,L)]\ln L}[2\pi\ln L\,\partial^2\tau/\partial q^{*2}]^{-1/2}\,. \tag{4.4c}$$

In passing from Eq. (4.4a) to (4.4b) we used $\alpha=-\partial\tau^{(0)}/\partial q^*$ and to get Eq. (4.4c) we dropped the term involving $\partial\tau^{(1)}/\partial q^*$, which leads to a correction of relative order $(\ln L)^{-2}$. The above is obviously merely the first term in a systematic expansion in powers of $(\ln L)^{-1}$. Thus, we obtain a result of the form of Eq. (2.9) but with

$$f(\alpha,L)=\frac{\ln P}{\ln L}=q^*\alpha+\tau(q^*,L)-\frac{1}{2\ln L}\ln[\partial^2\tau/\partial q^{*2}]\,. \tag{4.5a}$$

Here we dropped the term $\ln[2\pi\ln L]/\ln L$ since it represents an unimportant α-independent shift in $f(\alpha,L)$. Then

$$f(\alpha,L)=q^*\alpha+\tilde{\psi}(q^*)-\tilde{\psi}(0)+\frac{\ln A_{q^*}}{\ln L}-\frac{1}{2\ln L}[\ln\partial^2\tilde{\psi}(q^*)/\partial q^{*2}]\,. \tag{4.5b}$$

Thus, we have shown how finite-size effects can be incorporated systematically in powers of $1/(\ln L)$. We should emphasize that we have used Eq. (2.13) to define $f(\alpha,L)$. When fluctuations due to finite L are ignored, the saddle-point approximation ensures that $f(\alpha)$ and $\tau(q)$ are Legendre transforms of one another. However, when fluctuations due to finite-size effects are taken into account, then $f(\alpha,L)$ is no longer the Legendre transform of $\tau(q,L)$.

A related formulation was given some time ago[35] in the mathematical literature. That finite-size corrections are of order $(\ln L)^{-1}$ has been confirmed recently both by analytic calculations[36] for several special models and also by numerical methods.[16] Such corrections also follow directly in the explicit example presented in the Appendix. A further discussion of finite-size effects is given below in Sec. V C.

B. Alternate derivation of results for the moments of $\ln i^2$

In this subsection we give a derivation of Eq. (1.1) alternate to that of Sec. III. In contrast to the previous derivation, this one requires the stronger assumption that $f(\alpha)$ can be expanded in a power series in $(\alpha-\alpha_0)$ about its maximum at $\alpha=\alpha_0$. In this approach we determine the moments of the logarithms by

$$\mu_k=\int_{y_{\min}}^{y_{\max}}dy\,y^kP(y,L)\Big/\int_{y_{\min}}^{y_{\max}}dy\,P(y,L) \tag{4.6a}$$

$$\approx(\ln L)^k\int_{-\infty}^{\infty}d\alpha\,\alpha^k e^{f(\alpha,L)\ln L}\Big/\int_{-\infty}^{\infty}d\alpha\,e^{f(\alpha,L)\ln L} \tag{4.6b}$$

$$\equiv(\ln L)^k\frac{I_k}{I_0}\,, \tag{4.6c}$$

where I_k is the integral in the numerator. In Eq. (4.6a) $y_{\min}=-\ln i^2_{\max}$, where $i_{\max}$ is the current having the largest value not equal to unity. For $L\to\infty$, $i_{\max}\to 1$ and therefore $y_{\min}\to 0$. In going from Eq. (4.6a) to Eq. (4.6b) we assume that the integral is dominated by the region near the maximum of the integrand. Now we utilize the results of the previous subsection [particularly Eq. (4.5)] to include finite-size corrections to $f(\alpha)$. We have

$$I_k=\int d\alpha e^{f(\alpha,L)\ln L+k\ln\alpha} \tag{4.7a}$$

$$=(\alpha^*)^kL^{f(\alpha^*)}\int d\alpha\exp\left[\ln L\left[\tfrac{1}{2}(\alpha-\alpha^*)^2f_2+\tfrac{1}{6}(\alpha-\alpha^*)^3f_3+\cdots\right]+\frac{k}{\alpha^*}(\alpha-\alpha^*)-\frac{1}{2}k\frac{(\alpha-\alpha^*)^2}{(\alpha^*)^2}\cdots\right]\,, \tag{4.7b}$$

where $f_n\equiv d^nf/d\alpha^n|_{\alpha=\alpha^*}$ and α^* (which depends on L) is determined below by $f_1(\alpha^*,L)=0$. Since we are only interested in terms which are of order $(\ln L)^{-1}$ and which depend on k, we only need the expansions as written in Eq. (4.7b). Thus,

$$I_k\approx(\alpha^*)^kL^{f(\alpha^*)}\exp\left[-\frac{k^2}{2\alpha_0^2f_2(\alpha_0)\ln L}\right]\sqrt{\pi}\left[-\frac{1}{2}f_2(\alpha_0)\ln L+\frac{k}{2\alpha_0^2}\right]^{-1/2}\left[1+\frac{k}{6\alpha_0}f_3(\alpha_0)\ln L\frac{3}{[f_2(\alpha_0)\ln L]^2}\right]\,, \tag{4.8}$$

where again we neglected k-independent corrections and where we set $\alpha^*=\alpha_0$ in the correction terms. In the correction terms we may evaluate $f(\alpha,L)$ at $L=\infty$. Thus,

$$\mu_k=(\alpha^*\ln L)^k\left[1-\frac{k(k-1)}{2\alpha_0^2 f_2(\alpha_0)\ln L}+\frac{kf_3(\alpha_0)}{2\alpha_0 f_2(\alpha_0)^2\ln L}\right]. \tag{4.9}$$

In the correction terms we may use

$$f_2(\alpha_0)=-[\tilde{\psi}^{(2)}]^{-1}=\frac{dq}{d\alpha_0}, \tag{4.10a}$$

$$f_3(\alpha_0)=\tilde{\psi}^{(3)}/[\tilde{\psi}^{(2)}]^3. \tag{4.10b}$$

Also, since $\alpha^*-\alpha_0$ is of order $(\ln L)^{-1}$, Eq. (4.9) is

$$\mu_k=(\alpha_0\ln L)^k\left[1+\frac{k(k-1)\tilde{\psi}^{(2)}}{2\alpha_0^2\ln L}-\frac{k\tilde{\psi}^{(3)}}{2\alpha_0\tilde{\psi}^{(2)}\ln L}+\frac{k(\alpha^*-\alpha_0)}{\alpha_0}\right]. \tag{4.11}$$

To complete the calculation we need to evaluate $\alpha^*-\alpha_0$, where α^* is determined by $df(\alpha^*,L)/d\alpha^*=0$. So α^* is determined by

$$\frac{d}{d\alpha}\left[q^*\alpha+\tilde{\psi}(q^*)+\frac{\ln A_{q^*}}{\ln L}-\frac{1}{2\ln L}\ln\frac{\partial^2\tilde{\psi}(q^*)}{\partial q^{*2}}\right]_{\alpha=\alpha^*}=0, \tag{4.12}$$

where $q^*=q^*_\infty(\alpha)$ is determined by $\alpha=-\partial\tilde{\psi}(q^*)/\partial q$. When $\alpha=\alpha_0=-\partial\tilde{\psi}(q^*)/\partial q^*|_{q^*=0}$, then $q^*=0$, of course. From Eq. (4.11) α^* will deviate from α_0 by terms of order $(\ln L)^{-1}$, so q^* will be of order $(\ln L)^{-1}$. Evaluation of Eq. (4.12) gives

$$q^*_\infty(\alpha)+\frac{dq^*_\infty}{d\alpha}\left[\alpha+\frac{\partial\tilde{\psi}(q^*)}{\partial q^*}+\left[\left.\frac{\partial A_{q^*}}{\partial q^*}\right|_{q^*=0}-\frac{\tilde{\psi}^{(3)}}{2\tilde{\psi}^{(2)}}\right](\ln L)^{-1}\right]_{\alpha=\alpha^*}=0, \tag{4.13}$$

where we set $q^*=0$ in the correction terms and thus $A_0=1$. In other words

$$q^*_\infty(\alpha^*)=-\frac{dq^*_\infty}{d\alpha}\frac{1}{\ln L}\left[\left.\frac{\partial A_q}{\partial q}\right|_{q=0}-\frac{\tilde{\psi}^{(3)}}{2\tilde{\psi}^{(2)}}\right] \tag{4.14a}$$

$$=\frac{1}{\ln L}\left[\left.\frac{1}{\tilde{\psi}^{(2)}}\frac{\partial A_q}{\partial q}\right|_{q=0}-\frac{\tilde{\psi}^{(3)}}{2(\tilde{\psi}^{(2)})^2}\right]. \tag{4.14b}$$

Finally, expanding in powers of q^* we get

$$\alpha^*=-\frac{\tilde{\psi}(q^*)}{\partial q^*}=\tilde{\psi}^{(1)}-q^*\tilde{\psi}^{(2)} \tag{4.15a}$$

$$=\alpha_0+\frac{1}{\ln L}\left[-\left.\frac{\partial A_q}{\partial q}\right|_{q=0}+\frac{\tilde{\psi}^{(3)}}{2\tilde{\psi}^{(2)}}\right]. \tag{4.15b}$$

Substituting this into Eq. (4.11) indeed reproduces Eq. (3.5b).

C. Discussion: Analyticity near $q=0$

There is a very important distinction between the derivation presented in Sec. III and that of Sec. IV B. The first one works only for positive integral k, for which the cumulants can be defined and Eq. (3.4) holds. Since, as discussed after Eq. (3.4), we believe that for the percolating cluster (as distinct from special models proposed in the literature[14,31]), $\tilde{\psi}(q)$ is well behaved for all $q\geq 0^+$, we believe that Eq. (3.5) follows without further assumptions. The second derivation gives results which *a priori* hold for both positive and negative k. However, it assumes that μ_k is dominated by contributions from the integration over an interval of order $\Delta\alpha\sim(\ln L)^{-1/2}$ around the maximum of $f(\alpha)$ at α^*, in which it can be expanded as in Eq. (4.7b). This assumption may break down in two cases. First, the expansion of $f(\alpha)$ around $\alpha=\alpha_0$ may not converge over a range of α of order $\Delta\alpha\sim(\ln L)^{-1/2}$, or second, μ_k may be dominated by large contributions coming from outside this range. The first possibility requires that the radius of convergence of the expansion about α_0 is at most of order $(\ln L)^{-1/2}$. In such a case, it follows that in the "thermodynamic" limit, $L\to\infty$, $f(\alpha)$ becomes singular at $\alpha=\alpha_0$, implying that $\tilde{\psi}(q)$ becomes singular at $q_c=0$. The weakest singularity would arise if $f(\alpha)$ had different functional forms for $\alpha>\alpha_0$ and for $\alpha<\alpha_0$. For example, we might allow different expansion coefficients a_k^+ and a_k^- in Eq. (4.7b) for $\alpha<\alpha_0$ and $\alpha>\alpha_0$, respectively. Such differences yield corrections to μ_k of relative order $k(|a_1^-|^{-1}-|a_1^+|^{-1})/\sqrt{\ln L}$, which would thereby contradict Eq. (3.5). Since Eq. (3.5) follows from our direct (first) proof, such a scenario is excluded.

The second possibility may arise if $P(y)$ does not decay to zero sufficiently fast as y approaches the cutoffs $y_{\min}$ or $y_{\max}$. As we will discuss in Sec. V D, extremely large values of y arise from exponentially small currents. As we show below [Eq. (5.8)], the corresponding probability decays exponentially like

$$\ln P(y)\approx -by \tag{4.16a}$$

or

$$\hat{P}(i) \approx i^{2(b-1)} , \tag{4.16b}$$

which is equivalent to the linear behavior of the form

$$f(\alpha) = -b(\alpha - \alpha') , \tag{4.17}$$

where α' is related to the unspecified amplitude in Eq. (4.16b). It is easy to see that if Eq. (4.16a) holds over the interval $y_{\max} > y > y_0$, and even if $y_{\max}$ diverges as L^{ρ}, the contribution from this interval to the integrals in Eq. (4.6) will still be exponentially small compared to that of the region near α_0. Similar exponential decays occur for currents near unity, i.e., y near $y_{\min}$: even though $y_{\min} \sim L^{-x}$, resulting from chains of length L^x in parallel to a single bond, its probability is exponentially small. In both cases, the contributions from the integral boundaries will become negligible for sufficiently large L (and fixed k). Note, however, that Eq. (3.5) breaks down for $k \sim k_0$, where $|C_1 k_0 + D_1 k_0(k_0-1)/2| = \ln L$. In the limit of very large $|k|$, μ_k may be dominated by contributions from $y_{\min}$ (for $k<0$) or $y_{\max}$ (for $k>0$), and thus μ_k may exhibit exponential dependence on L. However, this behavior may arise only for k much larger than k_0, viz., for $k \geq y_{\max}/\ln y_{\max} \sim L^{\rho}/\ln L$.

If Eq. (2.2b) is valid for asymptotically large L, as implied by exact multifractality (and to within power-law corrections in $1/L$ according to the ϵ expansion), then all the coefficients $\tilde{\psi}^{(k)}$ and $f_k(\alpha_0)$ are universal and independent of L. Note that the Gaussian approximation, Eq. (2.6), is nothing but a truncation of the expansion Eq. (4.7b) at quadratic order, which is equivalent to a truncation of the expansion of $\tilde{\psi}(q)$ at the same order. Clearly, such truncations do very badly for higher moments and cumulants of y. They are even worse for M_q, where they violate[33,12] the approach of $\tilde{\psi}(q)$ to a finite limit as $q \to \infty$.

The expansion for $f(\alpha,L)$ in Eq. (4.7b) is expected to give a good description of $P(y,L)$ near its peak. Indeed, all the numerical curves of P versus y in the literature seem to have smooth maxima. However, these curves are very asymmetric, and one may need many terms, or one may need an alternative functional form, far away from the peak. Equation (4.7b) gives an infinite series expansion for $f(\alpha)$ near $\alpha = \alpha_0$. In contrast, some of the measured curves seem linear[16,17,19,34,37] for large α (small i). We discuss these data below in Sec. V E and in the Appendix. There exist many analytic functions $f(\alpha)$ which have a Taylor expansion like Eq. (4.7b) for α near α_0 and then become linear for large α, as in Eq. (4.17). A trivial example is $f(\alpha) = -\ln\cosh[b(\alpha-\alpha_0)]$, which starts like Eq. (4.7b) for small $(\alpha-\alpha_0)$ and approaches $f(\alpha) \approx -b\alpha$ for large α. In this example, the factors of $\ln L$ cancel and one has $\hat{P}(i^2) \sim i^{2b-2}$, as in Eq. (4.16b). An analytic approach of $f(\alpha)$ to a straight line will be shown below (Secs. V B and V C) to result from the ϵ expansion and related approximants. In other scenarios, Eq. (4.17) holds exactly over a range of finite α's, reflecting a "phase transition" in $f(\alpha)$ (See Sec. V D).

In conclusion, the validity of the alternative derivation of Eq. (3.5b) given in Sec. IV B implies that q_c is strictly negative and that, at least for $q > q_c$, $\ln P/\ln L$ approaches its asymptotic limit $f(\alpha)$ with finite-size corrections of relative order $1/(\ln L)$.

V. THE THRESHOLD q_c

In this section we consider the implications of various existing studies concerning the threshold q_c. Among these are the renormalization-group ϵ expansion (Sec. V B), numerical approximants to $\tilde{\psi}(q)$ (Sec. V C), rigorous bounds on q_c (Sec. V D), and numerical information concerning q_c (Sec. V E).

A. General comments

In the previous section we showed that the moments of $\ln i^2$ are not qualitatively affected by the "phase transition" at q_c, provided that q_c is strictly negative. We now return to a discussion of the moments of the currents, $M_q(L)$, and of the various possible scenarios for their behavior near and below q_c.

As already mentioned, q_c is identified as the value of q below which $\tau(q,L) = \ln M_q(L)/\ln L$ diverges to infinity as $L \to \infty$. For $q > q_c$, one has a well-defined finite limit

$$\lim_{L\to\infty} \tau(q,L) = \tilde{\psi}(q) - \tilde{\psi}(0) . \tag{5.1}$$

The different scenarios may now be classified according to how $\tilde{\psi}(q)$ approaches ∞ as q decreases through q_c, and how $\tau(q,L)$ varies with q and L for $q < q_c$.

In the simplest scenario, $\tilde{\psi}(q)$ diverges to ∞ as $q \to q_c^+$. Such behavior is predicted by the ϵ expansion, and is described in Secs. V B and V C, below. In this case, the slope $\alpha = -d\tilde{\psi}/dq$ also diverges as $q \to q_c^+$ and thus the Legendre transformation maps the range $q_c < q < \infty$ into the full range $0 < \alpha < \infty$ of $f(\alpha)$. The resulting function $f(\alpha)$ is completely analytic, and the existence of q_c is reflected in it only through its asymptotic slope $q = df/d\alpha$ as $\alpha \to \infty$. One might call this a "continuous phase transition." A similar situation occurs if $\tilde{\psi}(q)$ approaches a finite value at q_c^+, but with an infinite slope. This classification is equivalent to the usual one for phase transitions[38] of arbitrary order.

In a second scenario, $\tilde{\psi}(q)$ approaches a finite value, with a *finite* slope, α_c, as $q \to q_c^+$. In this case, the Legendre transform maps $\tilde{\psi}(q)$ (for $q > q_c$) onto a *finite* range of $f(\alpha)$, i.e., $0 < \alpha < \alpha_c$. The behavior of $f(\alpha,L)$ for $\alpha > \alpha_c$ then depends on details of the divergence of $\tau(q,L)$ as L increases for $q < q_c$. In Sec. V D we discuss several scenarios that may give rise to such behavior. We then finish this section with a critical review of available numerical information.

B. ϵ expansion and $d=6$

It is illuminating to start this discussion with the ϵ expansion of $\tilde{\psi}(q)$, for percolating networks in $d=6-\epsilon$ dimensions. Park, Harris, and Lubensky[13] found that, to leading order in ϵ,

$$\nu\tilde{\psi}(q) = 1 + \frac{a}{(q+1)(q+b^*)} , \tag{5.2}$$

with $a=\epsilon/14$ and $b^*=\frac{1}{2}$. Here, ν is the percolation correlation length exponent. We first note that the renormalization-group equations that lead to such an ϵ expansion in fact yield the more general form[40]

$$M_q(L) \propto \left[1+\frac{C}{\epsilon}(L^{\epsilon}-1)\right]^{\theta(q)}, \tag{5.3}$$

where C is a nonuniversal constant and

$$\theta(q)=\frac{1}{7(q+1)(q+1/2)}-\frac{2}{7}. \tag{5.4}$$

At $d=6$, this yields the exact result

$$M_q(L) \propto (\ln L)^{\theta(q)}. \tag{5.5}$$

Thus, multifracticality in L is replaced by multifractality in $\ln L$, with the nontrivial exact set of exponents $\theta(q)$. As far as we know, this kind of multifractality has not been identified before. Equation (5.4) yields a divergence of $\theta(q)$ as $q \to (-\frac{1}{2})^+$. Since M_q decreases monotonically[12] with q, this implies a breakdown of the simple power law of Eq. (5.5) for $q \le q_c = -\frac{1}{2}$. At least for $d=6$ it seems clear that $q_c=-\frac{1}{2}$ is definitely strictly negative. This may be interpreted by saying that for $q<q_c$, the moments $M_q(L)$ are dominated by very small currents, whose contribution grows with L faster than a power of $\ln L$.

Given Eq. (5.5), one may now follow the same algebra as in Secs. II and IV A and find a data collapse of $\ln P/\ln(\ln L)$ versus $y/\ln(\ln L)$, described by the Legendre transform of $\theta(q)$ for $q>-\frac{1}{2}$. As in the other case, it is not obvious what P does for large y. We hope these exact results will stimulate numerical simulations and direct derivations of the behavior in six dimensions.

C. Approximant for $d<6$

For finite but small ϵ, the $O(\epsilon)$ term in Eq. (5.2) also implies a divergence of $\tilde{\psi}(q)$ as $q \to -(\frac{1}{2})^+$. Since higher-order terms have not yet been calculated, we can foresee two possible scenarios: if the terms of order ϵ^k contain denominators of the form $(q+\frac{1}{2})^k$, with no poles at $q>-(\frac{1}{2})$, then one would conclude that the singularity in $\tilde{\psi}(q)$ occurs at $q_c=-(\frac{1}{2})+\mathcal{O}(\epsilon)$, moving continuously away from $-(\frac{1}{2})$. This would imply a finite negative value of q_c for a range of dimensions below $d=6$. If higher-order terms contain poles at smaller values of $-q$, then q_c may have a discontinuity from $-(\frac{1}{2})$ to some other value between $-(\frac{1}{2})$ and zero as d moves from 6 to $6-\epsilon$.

It is important to note that the recursion relations in the ϵ expansion are analytic because they describe the recursive removal of noncritical degrees of freedom.[41] Thus, at any finite order in ϵ we expect $\tilde{\psi}(q)$ to be a rational function of q. It may have poles (as a function of q), but we do not foresee the possibility of contributions to $\tilde{\psi}(q)$ of order q^{λ} with λ not an integer. Since a pole at $q=0$ is out of the question [$\tilde{\psi}(0)$ is the fractal dimension of the backbone], we expect derivatives of $\tilde{\psi}(q)$ to be finite as $q \to 0^+$. Specifically, we doubt that the ϵ expansion can produce terms of order q^{λ} which occur in the example given by Mandelbrot, Evertsz, and Hayakawa.[39] The point we emphasize is that the ϵ expansion gives rise to a much more restricted class of behavior than does multifractality in the broad sense used in Ref. 36. Finally, the ϵ expansion indicates that the leading corrections to the asymptotic behavior of $M_q(L)$ are of relative order L^{-x}, where $x=\epsilon+O(\epsilon^2)$. Such a correction is negligible compared to those of order $(\ln L)^{-1}$ considered here. Further non-power-law corrections not contained in the ϵ expansion are even less important.

In Ref. 12, BMAH used Eq. (5.2) as an approximant for $\tilde{\psi}(q)$ in general dimensions, and chose the parameters a and b^* to fit the known values of $\tilde{\psi}(0)$ (backbone) and $\tilde{\psi}(1)$ (resistance). The results gave excellent fits for the whole curve of $\tilde{\psi}(q)$ for $q \ge 0$, compared to series results.[12] Assuming that the functional form (5.2) is indeed correct for all $d<6$ [as it would if q_c has an ϵ expansion away from $-(\frac{1}{2})$], then $\tilde{\psi}(q)$ diverges to infinity as q approaches $q_c=-\min(b^*,1)$ from above.[42] BMAH estimated that $b^*=1.05\pm0.1$, 0.65 ± 0.08, 0.45 ± 0.1, and 0.33 ± 0.3 for $d=2$, 3, 4, and 5, respectively.

Since $\tilde{\psi}(q)$ is a monotonically decreasing function, a divergence in $\tilde{\psi}(q)$ as $q \to q_c^+$ implies that, for $q \le q_c$, one has $\tau(q,L) \to \infty$ as $L \to \infty$, as indeed happens if $M_q(L)$ grows with L faster than a power law (e.g., exponentially).[4,12] Using Eq. (5.2) with $a=1.22$ and $b^*=1.05$,[12] we have derived $\alpha(q)$ and $f(\alpha)$ for $d=2$ [see Eqs. (2.11) and (2.12)]. The results are shown in Fig. 2.[43] It is particularly interesting to note that $f(\alpha)$ [see Fig. 2(a)] looks linear for large α. The large values of α arise from negative values of q, with $\alpha \to \infty$ for $q \to q_c^+$. From Eqs. (2.11)

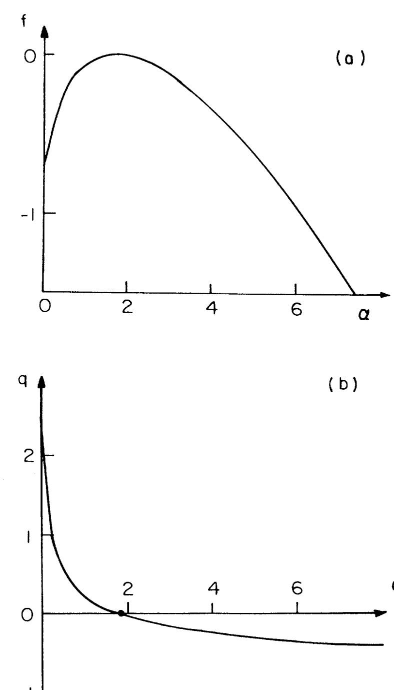

FIG. 2. Multifractal functions based on the approximant of Eq. (5.2) in two dimensions (after Ref. 12). (a) $f(\alpha)$. (b) $q(\alpha)$.

and (2.12), one has $df/d\alpha=q$. As q decreases from zero to q_c [see Fig. 2(b)], α grows from α_0 (where f has its maximum) to ∞, and $q=(df/d\alpha)$ changes very slowly from 0 to q_c [see Fig. 2(b)]. Thus the $f(\alpha)$ curve is locally very close to a straight line. For asymptotically large α (small currents), the slope approaches q_c and then one ends up with the asymptotic behavior of Eq. (4.17), with $b=-q_c$. However, we note that even when $\alpha=7$ in Fig. 2, i.e., when $i^2=L^{-7}$, the effective local slope of $f(\alpha)$, which we read from Fig. 2(b), is only of order $q\approx-0.4$, compared to $q_c=-\min(b^*,1)=-1$ here. Given a measured graph like Fig. 2(a), and fitting its right-hand side to an effective straight line, thus yields an upper bound for q_c which may be wrong (i.e., less negative) by more than a factor 2.

It should be noted that the above scenario yields a *continuous* function $f(\alpha)$ for *all* α, although $\tilde{\psi}(q)$ is infinite for $q<q_c$. Thus, the "phase transition" in $\tilde{\psi}(q)$ is not reflected by any singularity in $f(\alpha)$.

The specific form Eq. (5.2) may also be used to estimate the finite-size corrections to $f(\alpha)$, as given by Eq. (4.5b). We recall that the q dependence of A_q has been found numerically to be quite small.[15,20] Accordingly, the last term in Eq. (4.5b) shows that these corrections depend on $\partial^2\tilde{\psi}(q^*)/\partial q^{*2}$. From Eq. (5.2), this second derivative is of order unity for $q^*>0$, i.e., for $\alpha<\alpha_0$, but it diverges to infinity as $q^*\to q_c^+$. Thus, the finite-size deviations are expected to grow very rapidly as α grows from α_0 to ∞. Indeed, all the observed numerical estimates of $f(\alpha)$ show growing finite-size deviations from data collapse above the peak in $f(\alpha)$. See Sec. V C. We expect similar behavior in many other multifractal situations.[44]

D. Ladder configurations

Although the ϵ expansion yields a divergence of $\tilde{\psi}(q)$, it is not clear whether or not the field theory contains all the Griffiths-like rare small currents. Here we discuss their effect on $f(\alpha)$.

In Ref. 12, BMAH attributed the divergence of negative moments of the current to ladder configurations. They showed that a ladder with k rungs, whose minimal current for large k is of order

$$i(k)\sim\left[2+\sqrt{3}\right]^{-k}\equiv i_0^{-k}\ , \tag{5.6}$$

occurs with a probability which is at least as big as

$$p(k)\sim[\mu p^3(1-p)^{2z-6}]^k\equiv x^{-k}\ , \tag{5.7}$$

where μ is the branching ratio for self-avoiding walks on the dual lattice and z is the coordination number. Thus,

$$\hat{P}(i^2)=p(k)\frac{dk}{di^2}\sim i^{-2(q_{c_1}+1)}\ , \tag{5.8}$$

which is equivalent to Eq. (4.16) with

$$2b=-2q_{c1}=\frac{\ln x}{\ln i_0}\ . \tag{5.9}$$

For $d=2$, we set $z=4$, $\mu\approx3(2d-3)=3$, and $p=p_c=\frac{1}{2}$, so that $2q_{c_1}\approx-1.8$. Assuming that, for $q<q_{c_1}$, $i(k)$ dominates the moment M_q, and assuming that for a cluster of size L one has $k\propto L^{\rho}$, one ends up with Eq. (2.3a).

In fact, BMAH used such ladder configurations to prove that Eq. (2.3a) holds for $q<q_{c1}$, where a lower bound (which was negative) for q_{c_1} was given. BMAH also estimated q_{c1} from series expansions for the concentration dependence of the cluster averages of $M_q(\mathbf{x},\mathbf{x}')$. These averages diverged at a threshold p_c which became q dependent for $q<q_{c1}$, with $q_{c1}\approx-0.3$ and -0.6 for $d=2$ and 3, respectively, indicating an exponential decrease of the small currents. [To see this, note that for $p_c(q)<p_c$ the probability that a cluster of s sites occurs is of order[24] $\exp(-cs)$, where $c>0$ for $p_c(q)\neq p_c$. To have a divergent $M_q(L)$ for $L\to\infty$ at $p_c(q)<p_c$ thus requires that $i_s^{2q}\sim\exp(cs)$, where i_s is the minimum current in the cluster of s sites.] As stated above, the thresholds q_{c1} and q_c may differ from each other if the network contains currents which have some behavior intermediate between a power law and Eq. (2.3a), as in Eq. (2.3b).

At least in two dimensions, numerical work does tend to suggest that q_c and q_{c1} are indeed different. In particular, note the estimate given after Eq. (5.9), $q_c\approx-0.9$, which is surprisingly close to the BMAH approximant value of $q_c=-1$. In contrast, the work of Batrouni, Hansen, and Roux[16] indicates that the limiting (for $\alpha\to\infty$) slope $df/d\alpha=-q_c$ becomes very small as $L\to\infty$. Their small currents (at large α) are probably not due to ladders, both because the associated value of q_c is not as expected and also because to see ladders would require an astronomically large number of trials.

Accordingly, it is of interest to see qualitatively how $f(\alpha,L)$ behaves for large α. To see the effect on $f(\alpha,L)$ of ladders in more detail, we need to estimate the prefactors in Eqs. (5.6) and (5.7). We write

$$i(k)\sim L^{-\alpha_0/2}i_0^{-k}\ , \tag{5.10a}$$

$$p(k)\sim x^{-k}L^{-h}\ . \tag{5.10b}$$

Equation (5.10a) expresses the fact that a ladder will most likely be attached in parallel with the most probable current, $i_b^2\sim L^{-\alpha_0}$. It is difficult to estimate the probability of having a ladder of k links. The presence of the factor x^{-k} is clear. However, the prefactor takes proper account of the fact that we wish to consider only ladders which are part of the network connecting two sites separated by a distance of order L. This constraint probably introduces a power-law prefactor as written in Eq. (5.10b). We do not estimate h, although we believe it to be non-negative. Using Eq. (5.10) we have

$$P(y)=p(k)\frac{dk}{dy} \tag{5.11a}$$

$$\sim e^{\ln x[(\ln i/\ln i_0)+(\alpha_0/2)(\ln L/\ln i_0)]}L^{-h}(2\ln i_0)^{-1}\ . \tag{5.11b}$$

Therefore,

$$f(\alpha)=\frac{\ln P(y)}{\ln L}\sim-b(\alpha-\alpha_0)-h\ , \tag{5.12}$$

with $b = \ln x/(2\ln i_0) = -q_{c1}$. For finite L, we can apply this result up to a maximum value of α, given by $\alpha_{\max} \sim L^{\rho}/\ln L$, which for our purposes is infinite, since it is larger than any other relevant quantities. For α only somewhat larger than α_0, Eq. (5.12) only gives a lower bound on $f(\alpha)$: currents corresponding to such values of α may arise on many other bonds, and not only on ladders. The actual $f(\alpha)$ will thus be above the straight line of Eq. (5.12). However, for any finite L there exists a value $\alpha_x(L)$ such that for $\alpha > \alpha_x(L)$ one remains practically only with the ladder currents. This crossover value $\alpha_x(L)$ is expected to grow as L increases. This reasoning yields the scenario plotted schematically in Fig. 3: For any finite L, $f(\alpha,L)$ decreases slowly up to $\alpha \approx \alpha_x(L)$, and then drops to the dashed line, Eq. (5.12), corresponding to the ladders (drawn assuming $h=0$). The slope of this dashed line, q_{c1}, may be much larger than that of $f(\alpha,L)$ for $\alpha < \alpha_x(L)$. As $L \to \infty$, $\alpha_x(L) \to \infty$, and the asymptotic slope of $f(\alpha, L\to\infty)$, q_c, may, in fact, be smaller in magnitude than $|q_{c1}|$. This asymptotic $f(\alpha)$ curve is shown by the dot-dashed line in Fig. 3. In summary, in this scenario, the ladders do not affect the asymptotic $f(\alpha)$.

The scenario of Fig. 3 can be discussed in terms of the behavior of $\tau(q,L)$. As Fig. 3 implies, the ladder configurations dominate the current distribution for sufficiently small i, say for $i < i_1(L)$. In this range we expect Eq. (4.16) to hold, with $b = -q_{c1}$. We write

$$M_q(L) = I_1 + I_2 \tag{5.13}$$

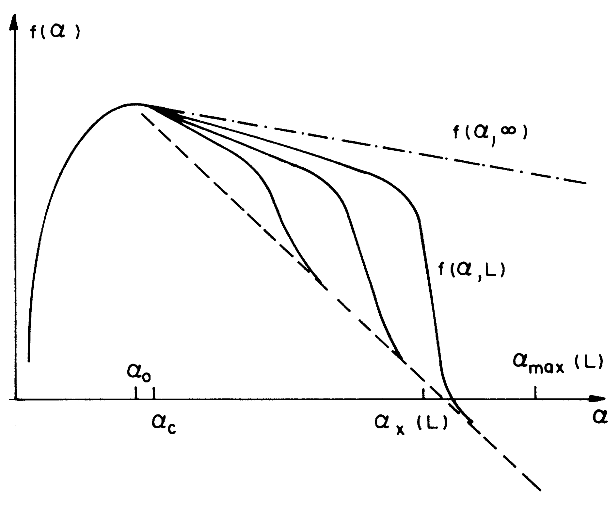

FIG. 3. Scenario for $f(\alpha,L)$ for three large L's and $\alpha > \alpha_0$. The qualitative points we stress here are as follows. (a) As L increases, the quasilinear part of $f(\alpha,L)$ evolves into the asymptotic $f(\alpha)$ (dash-dotted line) which has a small slope (according to Ref. 16). (b) For any finite L, $f(\alpha,L)$ must eventually coincide with the asymptotic result for the exponentially small currents in the ladder configurations (Ref. 12) (dashed line). This "ladder" line ends at $\alpha_{\max} \sim L^{\rho}/\ln L$. (c) The crossover to the ladders occurs at a value of α, $\alpha_x(L)$, which becomes infinite as $L\to\infty$, so that asymptotically the ladders do not affect $f(\alpha)$. (d) If there is a phase transition to an exactly linear $f(\alpha)$, it occurs at $\alpha > \alpha_0$.

with

$$I_1 = \int_{y_{\min}}^{y_1} dy\, P(y,L)\exp(-qy) , \tag{5.14}$$

$$I_2 = \int_{y_1}^{y_{\max}} dy \exp[(q_{c1}-q)y] \tag{5.15a}$$

$$= \{\exp[(q_{c1}-q)y_{\max}] - \exp[(q_{c1}-q)y_1]\}/(q_{c1}-q) , \tag{5.15b}$$

where $y_1(L) = -\ln i_1^2(L)$ and where we used Eq. (4.16) in Eq. (5.15a). It is now easy to see that I_2 is negligibly small for $q > q_{c1}$, where $M_q(L)$ is dominated by I_1. However, for $q < q_{c1}$, I_2 is dominated by its upper cutoff, so that

$$I_2 \propto \exp[(q_{c1}-q)y_{\max}] \propto (i_{\min}^2)^{(q-q_{c1})} . \tag{5.16}$$

Since $i_{\min}$ decays faster than a power of L, I_2 will dominate M_q and we find

$$\tau(q,L) = (q_{c1}-q)y_{\max}/\ln L \sim (q_{c1}-q)L^{\rho}/\ln L . \tag{5.17}$$

Thus, $\tau(q,L)$ is *linear* in q, with a slope that diverges to ∞ with L.

E. Numerical information

Now we review briefly information available from existing numerical studies of the distribution of currents. Most of these studies exhibit an apparent straight line for $f(\alpha)$ at large α, as in Eq. (4.17). The corresponding asymptotic slopes are presented in Table I and are discussed below. We remind the reader that a linear result of the form of Eq. (4.17) would imply that $q_c = -b$. With the exceptions noted below, the results were obtained as follows. The system studied was an $L \times L$ (or for the three-dimensional work[20] $L\times L\times L$) system across which a potential difference was imposed. Then the distribution of currents $\hat{P}(i)$, or equivalently $P(-\ln i^2)$, was obtained by an ensemble average. Kahng's $L=4$ result[19] was obtained by an *exact* enumeration of the configurational average for a system of 4×4 sites. Straley's results[17] were obtained by imposing an electric field throughout the

TABLE I. Values of the asymptotic slope of $f(\alpha)$, b for the current distribution on a percolation cluster.

Ref.	L	b [Eq. (4.17)]	$b \ln L$
		Two dimensions	
33	90	0.255	1.15
19	4	1.15	1.59
17	10	0.48	1.1
17	20	0.38	1.14
17	40	0.31	1.14
17	80	0.26	1.14
16	32	0.31	1.07
16	64	0.25	1.04
16	128	0.18	0.87
		Three dimensions	
20	18	0.8	2.3

sample. His results displayed the expected dependence on the orientation of the electric field.

The results of these simulations are in broad agreement with one another. For α less than the value α_0, for which $f(\alpha)$ is maximal, $df(\alpha)/d\alpha$ is large (except when α is very near α_0) and partly for this reason data collapse is observed, i.e., $f(\alpha)$ does not display noticeable dependence on L. For $\alpha>\alpha_0$ (but note that the sign convention for α in Ref. 16 differs from the standard one used by other authors), $f(\alpha)$ has significant dependence on L. Very crudely, as noted in Ref. 16, for $\alpha>\alpha_c>\alpha_0$, one may describe $f(\alpha,L)$ by

$$f(\alpha,L)\approx f(\alpha_c)-b(L)(\alpha-\alpha_c)\ , \tag{5.18}$$

with a slope $b(L)$ which decreases as L increases. The data of Ref. 16 were roughly described by

$$b=a/\ln L\ , \tag{5.19a}$$

which would imply that $q_c=b(\infty)=0$. Such a conclusion disagrees with our arguments given in Secs. III and IV. Moreover, the data interpretation was based on the assumption that f could not become negative. Dropping that assumption yields an asymptotic estimate $b_\infty\approx\frac{1}{4}$ as $L\to\infty$ (A. Hansen, private communication). We therefore think that probably a better description of the results would involve setting

$$b=b_\infty+a/\ln L\ . \tag{5.19b}$$

Table I presents values of $b\ \ln L$, which would be independent of L if Eq. (5.19a) were exactly correct. The numerical results are not really accurate enough to distinguish between the two possibilities of Eqs. (5.19a) and (5.19b), much less to determine the true corrections of order $(\ln L)^{-1}$, given in Eq. (4.5b), which need not be a simple linear function of α as assumed by both Eqs. (5.19a) and (5.19b). In fact, examination of the curves of Batrouni, Hansen, and Roux[16] shows that not only do neither of these two equations fit the asymptotic slope very well, but also $f(\alpha)$ has some curvature which confirms that the finite-size corrections are not simply linear functions of α. Indeed, there are regions of $f(\alpha)$ for finite L for which $\partial^2 f/\partial\alpha^2>0$, a situation which is excluded for $L\to\infty$, but which may arise for finite L (see Fig. 3). Similar "anomalous" finite-size corrections have been analyzed by Mandelbrot[36] and co-workers[39] for special models of multifractality.

In summary, however, with the above caveats, one may say that the data are described by Eq. (5.19a) or by Eq. (5.19b) with a small (possibly zero) value of b_∞ and with $a\approx 1.1\pm0.05$ in two dimensions. Unfortunately, the numerical difficulties prevent meaningful determinations at higher spatial dimensions where the prediction from the ϵ expansion that $q_c\neq 0$ might be tested. In fact, data at $d=5$ and 6, might provide crucial corroboration of our assertion that $q_c\neq 0$.

VI. CONCLUSIONS

In this paper we have studied the distribution of currents in the random diluted resistor network at the percolation threshold when a unit current is inserted at one terminal and removed at another at a separation of order L in a sample of linear size of order L. Our main conclusions are as follows.

(1) For $\ln i/\ln L$ finite, the distribution of currents is a function of the variable $\ln i^2/(\ln L)^\phi$, with $\phi=1$. In terms of this variable the distribution is thus essentially unifractal.

(2) In Sec. III, by considering the relation between the moments of the currents and the *cumulant* moments of the logarithms of the currents, we found that the assumption of power-law scaling for the former leads to the prediction for the latter (denoted μ_k^c):

$$\mu_k^c\sim\tilde{\psi}^{(k)}\ln L\ , \tag{6.1a}$$

where the $\tilde{\psi}^{(k)}$'s are universal constants which are derivatives of the so-called noise exponents for the random resistor network. This striking result can then be used to obtain the moments of the logarithms of the currents (denoted μ_k) to be

$$\mu_k\sim(\alpha_0\ln L)^k+O[(\ln L)^{k-1}]\ , \tag{6.1b}$$

with $\alpha_0=\tilde{\psi}^{(1)}$. These predictions are consistent with series work [see paper II (Ref. 25)] and existing transfer matrix data of Duering *et al.*[34] These results should hold for spatial dimensionality, d, greater than one and smaller than six. (Results for $d=1$ are given elsewhere.[25])

(3) Further to point (2): one can construct a family of ratios of the form $\mu_k\mu_l/(\mu_m\mu_n)$, where $k+l=m+n$, whose asymptotic value is predicted to be unity.

(4) In Sec. IV, we discussed the corrections to scaling of Eq. (6.1). In particular, we found corrections, given in Eq. (3.5), in the variables $k/\ln L$ and $k^2/(\ln L)$. Since these corrections are linear in $(\ln L)^{-1}$, they impose severe practical limitations on the accuracy with which Eqs. (6.1) can be verified numerically. They also imply similar finite-size corrections to scaling for other systems, such as DLA, viscous fingering, or dielectric breakdown, which are described by the multifractal picture.[44]

(5) Under the assumption that the multifractal function $f(\alpha)$ is analytic near its maximum, or equivalently, that the qth moment of the current distribution is analytic in q near $q=0$, we presented (in Sec. IV) an alternative analysis of the finite-size corrections of order $(\ln L)^{-1}$. The results so obtained for $f(\alpha)$ agree with those given in Eq. (3.5) obtained from the cumulant moments, μ_k^c, and with those found previously in the mathematical literature.[35] The fact that this alternative analysis of the finite-size corrections agrees exactly with that based on a study of the cumulants lends support to the assumption that, q_c, the critical value of q where the analyticity of the moments breaks down, is *not* zero. The form of the finite-size corrections we find is consistent with some model analytic[35,36] work.

(6) There exists a value of q_c such that the qth moment of the current distribution scales with a power law $\tilde{\psi}(q)$ which diverges for $q<q_c$. This fact indicates that $\partial f/\partial\alpha\to q_c$ as $\alpha\to\infty$. A consequence of this fact is that the distribution function for currents has an asymptotic power-law form at small currents:

$$\hat{P}(i^2)\sim i^{-2(q_c+1)}\ . \tag{6.2}$$

(7) Both the ϵ expansion and our analyticity arguments suggest that q_c is strictly negative: $q_c < 0$. Numerical data are not yet sufficiently comprehensive to confirm or refute this assertion.

(8) We showed (in Fig. 3) a scenario for the evolution with L of $f(\alpha,L)$ which incorporates (a) the analytic structure required by exponentially small currents in ladder configurations[12] and (b) the numerical studies of Batrouni, Hansen, and Roux[16] which for large L show a small slope for $df/d\alpha$ in the limit of large α.

(9) At $d=6$, we find the exact result that

$$M_q(L) \sim (\ln L)^{\theta(q)} , \tag{6.3}$$

with $\theta(q)$ given by Eq. (5.4).

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APPENDIX: EXACT RESULTS FOR HIERARCHICAL MODELS

We start with the MGF shown in Fig. 1. Similar results hold for a general family of randomized hierarchical structures.[33] The MGF has been found to describe qualitatively many properties of the spanning cluster for two-dimensional percolation.[22] A unit current is injected into one end and extracted from the other end of the structure. After N iterations, the currents in the bonds are given by

$$i(j,l) = (\tfrac{1}{4})^j (\tfrac{3}{4})^l , \tag{A1}$$

where $j,l = 0,1,\ldots,N$ and $j+l \le N$. The number of bonds carrying exactly this current is

$$n(j,l) = \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} N-j \\ l \end{bmatrix} 2^{N-j-l} 3^j , \tag{A2}$$

and the total number of backbone bonds is 6^N. The linear size scales as $L = 3^N$. Thus,

$$\begin{aligned} M_q &= 6^{-N} \sum_{j,l=0}^{N} n(j,l) i(j,l)^{2q} \\ &= \{[2+3(\tfrac{1}{4})^{2q} + (\tfrac{3}{4})^{2q}]/6\}^N , \end{aligned} \tag{A3}$$

and

$$\tilde{\psi}(q) - \tilde{\psi}(0) = \{\ln[2+3(\tfrac{1}{4})^{2q}+(\tfrac{3}{4})^{2q}] - \ln 6\}/\ln 3 . \tag{A4}$$

Using Eqs. (3.1) and (3.2), it is now straightforward to derive μ_k and μ_k^c. We find

$$\mu_1^c = \mu_1 = -[\tfrac{1}{3}\ln(\tfrac{3}{4}) + \ln(\tfrac{1}{4})]N \approx 1.482N , \tag{A5a}$$

$$\begin{aligned} \mu_2^c = \{ &\tfrac{2}{3}[\ln(\tfrac{3}{4})]^2 + 2[\ln(\tfrac{1}{4})]^2 \\ &- [\tfrac{1}{3}\ln(\tfrac{3}{4}) + \ln(\tfrac{1}{4})]^2\}N \approx 1.702N , \end{aligned} \tag{A5b}$$

$\mu_3^c \approx -0.1354N$, and $\mu_4^c \approx -5.581N$. From this we find

$$\mu_k \approx (1.482N)^k [1 + 0.387k(k-1)/N + O(N^{-2})] . \tag{A6}$$

Since $N \sim \ln L$, this result is of the form of Eq. (3.5). Equations (A5) and (A6) also result from a direct averaging over $|\ln i(j,l)^2|^k$ with the weights of Eq. (A2), confirming the use of Eqs. (3.2) and (3.3). Note that the above asymptotic results hold only for $k(k-1) \ll N \sim \ln L$. We will not consider the opposite limit when $k(k-1) \gg N$.

The numerical values in Eq. (A5) may be used to obtain the expansion, Eq. (4.7b), as an approximant for $\ln P$ near its maximum. For large α (small currents), P is dominated by Eq. (A2), with j near N and l near zero. Concentrating only on the points with $l=0$, and assuming $1 \ll N-j \ll N$ so that

$$\ln \begin{bmatrix} N \\ j \end{bmatrix} \approx (N-j)\ln N ,$$

we have

$$\begin{aligned} \ln n(i) &\sim C(N) - j[\ln N + \ln 2 - \ln 3] \\ &= C(N) + \gamma \ln i \end{aligned} \tag{A7}$$

with

$$\gamma \approx \frac{\ln N + \ln 2 - \ln 3}{\ln 4} . \tag{A8}$$

Since the effective slope γ depends only very weakly on L, via $\ln N \sim \ln(\ln L)$, data on a finite range of sizes might mislead one to conclude that the slope γ approaches a constant value, and therefore that there exists a finite negative threshold q_c. Indeed, for $40 < N < 320$, Eq. (A8) yields $2.4 \le \gamma \le 3.9$, in rough agreement with recent numerical plots of a coarse grained version[37] of $\ln n$, as given in Eq. (A2). However, we note that Eq. (A7) is only a rough approximation, and that $\ln n$ is never really linear in $\ln i^2$, so that, in fact, $q_c = -\infty$. As we discussed, for the percolating cluster, linearity at large α results from the exponentially small currents, which are not included in the hierarchical model. Therefore, we believe that hierarchical models completely miss the dominant effects of small currents. However, they are useful to illustrate the other features discussed in the present paper.

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