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LETTER TO THE EDITOR

Infinite set of exponents describing physics on fractal networks

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Abstract. The generalised resistance between connected points a distance L apart on fractal networks of non-linear ($V \sim I^{\alpha}$) resistors scales as $L^{\tilde{\zeta}(\alpha)}$. We show that $\tilde{\zeta}(\alpha)$ for $\alpha = -\infty$, -1, 0^-, 0^+, 1 and ∞, describes physically relevant geometrical properties and $\mathrm{d}\tilde{\zeta}/\mathrm{d}\alpha \leqslant 0$. For percolating clusters we give approximants for $\tilde{\zeta}$ for $-\infty < \alpha < \infty$ in 2-6 dimensions. For $\alpha < 0$ a family of solutions to Kirchhoff's equations exists, reminiscent of metastable states in spin glasses.

Random resistor fractal networks have been the subject of much fruitful research recently. Different properties of such systems are found to probe different critical exponents, or fractal dimensionalities. Choosing two terminals a Euclidean distance L apart, and putting a voltage V between them, generates a current $I = V/R$, and the resistance scales as $R(L) \sim L^{\tilde{\zeta}_R}$ (the resistance of each bond is $r = 1$). Current flows only via *backbone* bonds, whose number M_B scales with the fractal dimensionality of the backbone D_B: $M_B \sim L^{D_B}$. The characterisation of the network also involves the L_{SC}, *singly connected bonds*, which carry the full current I, with $L_{SC} \sim L^{\tilde{\zeta}_{SC}}$ (these determine the low temperature spin correlation functions (Coniglio 1981, 1982, Aharony *et al* 1984) and give bounds for the elastic response of networks (Kantor and Webman 1984)), the length of the minimal path between the terminals $L_{\min} \sim L^{\tilde{\zeta}_{\min}}$ (related to the rate of propagation of a flow front through the cluster (Stanley 1986)) and other physically relevant subgroups described below.

The study of percolating resistor networks recently led to the identification of several infinite sets of exponents, relevant to their physical properties. Rammal *et al* (1985) considered resistance fluctuations arising from microscopic noise in each resistor r_i in the network. The kth cumulant of $R(L)$ then scales as $\langle R^k \rangle_c \sim L^{x(k)}$. If the current in the ith bond is $I_i = Ii_i$, then (Rammal *et al* 1985)

$$\langle R^k \rangle_c = \sum_i \langle r_i^k \rangle_c |i_i|^{2k} \tag{1}$$

so that $\langle R^k \rangle_c$ is related to the $2k$th moment of the current distribution studied by de Arcangelis *et al* (1985b). The exponents $x(k)$ are very similar to the infinite set of

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exponents introduced earlier by Mandelbrot (1974) (see also Hentschel and Proccacia 1983) to describe the *fractal measures* of probability distributions on fractals and used to characterise intermittent turbulence (Mandelbrot 1978) and growth of diffusion-limited aggregates (Halsey *et al* 1986).

A different infinite set of exponents arises from a network of elements each of which obeys a non-linear generalisation of Ohm's law (Kenkel and Straley 1982): $V_i = r_i|I_i|^{\alpha}\operatorname{sgn} I_i$. Blumenfeld and Aharony (1985) showed that the resistance $R_\alpha(L) = |V|/|I|^{\alpha}$ now scales as $R_\alpha(L) \sim L^{\tilde{\zeta}(\alpha)}$, and identified $\tilde{\zeta}(\infty) = \tilde{\zeta}_{SC}$, $\tilde{\zeta}(1) = \tilde{\zeta}_R$ and $\tilde{\zeta}(0^+) = \tilde{\zeta}_{min}$. The fact that the single unifying function $\tilde{\zeta}(\alpha)$ interpolates between several relevant exponents generates interest in the general functional properties of $\tilde{\zeta}(\alpha)$, in possible relations between $\tilde{\zeta}(\alpha)$ and other sets (e.g. $x(k)$), and in possible additional useful values of α. The present letter addresses these questions.

Our new results are summarised as follows.

(a) We generalise the study of non-linear resistor networks to $\alpha < 0$, where new phenomena appear. In this regime there exists in general a family of solutions to Kirchhoff's (generalised) equations, corresponding to different directions of the currents through some bonds. (An example of a network with three solutions is shown in figure 1.) Each of these solutions corresponds to a local extremum in the power, P, as a function of the node voltages, where

$$P = \sum_i r_i|I_i|^{\alpha+1} = R|I|^{\alpha+1}. \tag{2}$$

These extrema are reminiscent of the metastable states in an Ising spin glass. Unless otherwise stated, we calculate $\tilde{\zeta}$ using the solution with the largest P.

(b) All these solutions give the same value of $\tilde{\zeta}$ at $\alpha = -1$, $\tilde{\zeta}(-1) = D_B$.

(c) Unless the network is fully *symmetric* (de Arcangelis *et al* 1985a) (e.g. $L_2 = L_3$ in figure 2), $\tilde{\zeta}(\alpha)$ has a discontinuity at $\alpha = 0$. The length, L_{max}, of the *longest* self-avoiding walk (SAW) between the two terminals scales as $L_{max} \sim L^{\tilde{\zeta}_{max}}$, with $\tilde{\zeta}_{max} = \tilde{\zeta}(0^-)$.

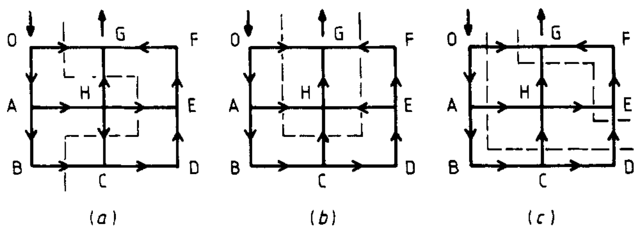

Figure 1. A cluster with source at O and sink at G for which three different solutions, (*a*), (*b*), (*c*), to Kirchhoff's equations exist. Arrows indicate directions of current and broken lines a cutting surface. In (*c*) two possible choices of cutting surfaces are shown.

(d) The relation $\tilde{\zeta}(\alpha) = x[(\alpha+1)/2]$ found by de Arcangelis *et al* (1985a) for the *fully symmetric* case, breaks down in the more general case, as the ε-expansion results for $x(k)$ (Park *et al* 1986) and $\tilde{\zeta}(\alpha)$ (Harris 1986) indicate. However, it always holds at $\alpha = \infty$, 1 and -1.

(e) For $\alpha \to -\infty$, $\tilde{\zeta}(\alpha) \sim z|\alpha| + O(1)$. The parameter z describes the scaling of the maximal 'cutting surface' of the backbone between the terminals, i.e. the largest number of bonds, N_{max}, which one can cut in order to break the backbone into two pieces, each connected to one terminal: $N_{max} \sim L^z$.

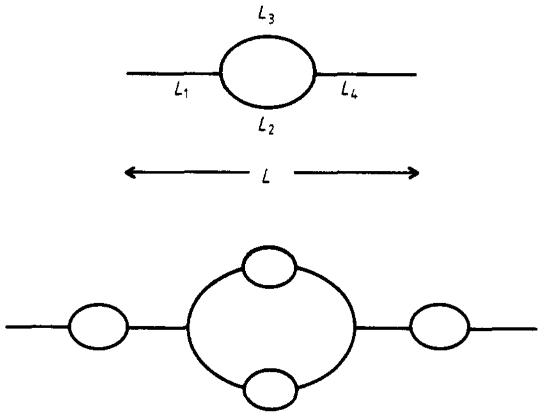

Figure 2. Two stages of iteration of the hierarchical model for the backbone of the percolation cluster.

(f) The function $\tilde{\zeta}(\alpha)$ is monotonically decreasing, $\tilde{\zeta}' \equiv \mathrm{d}\,\tilde{\zeta}(\alpha)/\mathrm{d}\alpha \leqslant 0$. Equality holds at $\alpha = \infty$. Unless the network is symmetric on all length scales, $\tilde{\zeta}'$ also vanishes for $\alpha \to 0^+$ and $\alpha \to 0^-$. This immediately proves the inequalities $D_{\mathrm{B}} \geqslant \tilde{\zeta}_{\max} \geqslant \tilde{\zeta}_{\min} \geqslant \tilde{\zeta}_R \geqslant \tilde{\zeta}_{\mathrm{SC}}$, without appealing to any specific geometrical model.

(g) The average length of a SAW on the cluster scales as $L_{\mathrm{SAW}} \sim L^{1/\nu_{\mathrm{SAW}}}$ (Kremer 1981). Since $\tilde{\zeta}_{\max} \geqslant 1/\nu_{\mathrm{SAW}} \geqslant \tilde{\zeta}_{\min}$, our results put strong bounds on ν_{SAW}.

(h) To generalise to the non-linear case replace the exponent $2k$ in (1) by $(\alpha+1)k$, and $x(k)$ by $\tilde{\psi}(\alpha, k)$, so that $\tilde{\psi}(1, k) \equiv x(k)$ and $\tilde{\psi}(\alpha, 1) \equiv \tilde{\zeta}(\alpha)$. For either (i) $k = 1$ and all α†, or (ii) all k in the limit $\alpha \to -1$ we prove that

$$(\partial\tilde{\psi}/\partial\alpha)/k = (\partial\tilde{\psi}/\partial k)/(\alpha+1). \tag{3}$$

(i) In view of all the above information, we constructed the following approximant function for percolation clusters in spatial dimension d at p_c:

$$\zeta(\alpha) \equiv \tilde{\zeta}(\alpha)\nu = 1 + \ln[1 + a(1 + b^{-1/\alpha})^{-c\alpha}] \tag{4}$$

where $b > 1$, a and c are parameters and ν is the exponent of the percolation correlation length, $\xi \sim |p - p_c|^{-\nu}$. This function satisfies Coniglio's (1981) theorem, that $\zeta(\infty) = \tilde{\zeta}_{\mathrm{SC}}\nu = 1$. Since the values of $\zeta(-1) = D_{\mathrm{B}}\nu$, $\zeta(0^+) = \tilde{\zeta}_{\min}\nu$ and $\zeta(1) = \tilde{\zeta}_R\nu$ are known with relatively high accuracy, we used them to determine a, b and c. For $d = 2$ and 3 this approximant is compared with series evaluations (MBAH) of $\zeta(\alpha)$ in figure 3, and the agreement is excellent. This approximant leads to the following estimates for the quantities $(-\nu\tilde{\zeta}'\ (\alpha = 1),\ \zeta(0^-) = \tilde{\zeta}_{\max}\nu,\ z\nu)$: for $d = 2$: $(0.22 \pm 0.01, 1.77 \pm 0.03, 0.89 \pm 0.01)$; for $d = 3$: $(0.09 \pm 0.03, 1.4 \pm 0.1, 0.88 \pm 0.07)$; for $d = 4$: $(0.05 \pm 0.02, 1.16 \pm 0.03, 0.98 \pm 0.02)$; for $d = 5$: $(0.02 \pm 0.02, 1.1 \pm 0.5, 1.0 \pm 0.3)$; for $d = 6 - \varepsilon$: $(0.02\varepsilon, 1 + 0.095\varepsilon, 0.8)$. For $d > 6$, $\zeta(\alpha) = 1$ for all α. The values for $\nu\tilde{\zeta}'\ (\alpha = 1)$ agree within the error bars with those we directly extracted from our series values, and with Harris's (1986) $\nu\tilde{\zeta}'\ (\alpha = 1) = -\varepsilon/72$. Our estimate for $\zeta(0^-)$ for $d = 2$ agrees within the error bars with $\tilde{\zeta}_{\max}\nu = 1.84$, found by Hong and Stanley (1983). Interestingly, existing estimates (Kremer 1981) for $1/\nu_{\mathrm{SAW}}$ are very close to our $\tilde{\zeta}_{\max}$.

† The case $k = 1$ (all α) was also shown by Harris (1986). For this case Meir *et al* (1986, hereafter referred to as MBAH) have also confirmed (3) using low concentration series.

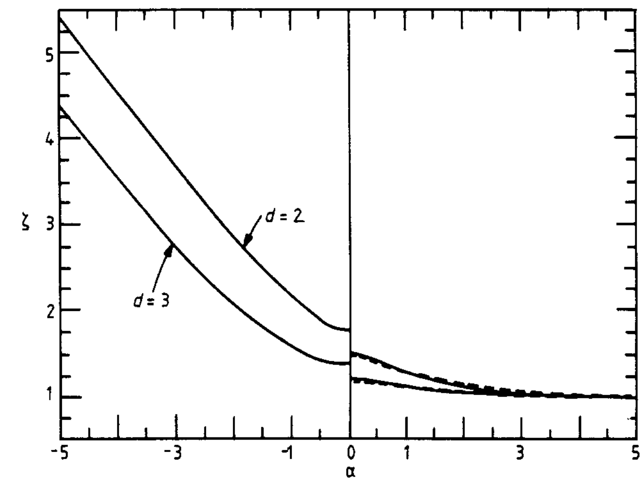

Figure 3. Approximant of (4) (full curve), compared (for $\alpha > 0$) with series results (MBAH) for ζ (broken curve).

We illustrate our results on the hierarchical fractal model shown in figure 2. For $L_1 = L_2 = L_4 = 1$ and $L_3 = 3$, this model reduces to that of Mandelbrot and Given (1984), designed to describe two-dimensional percolation. De Arcangelis *et al* (1985a) considered the symmetric case with $L_2 = L_3$. Following Blumenfeld and Aharony (1985), one can show that

$$R = L_{SC} + L_2[1 + (L_3/L_2)^{-1/\alpha}]^{-\alpha} \tag{5}$$

with $L_{SC} = L_1 + L_4$. For the asymmetric case, $L_3 > L_2$, we find $R = L_{SC} + L_2 = L_{\min}$ if $\alpha \to 0^+$ and $R = L_{SC} + L_3 = L_{\max}$ if $\alpha \to 0^-$, and $\mathrm{d}R/\mathrm{d}\alpha = 0$ in both limits. The two limits become identical, with $\mathrm{d}R/\mathrm{d}\alpha < 0$, only in the symmetric case, $L_2 = L_3$. For $\alpha = -1$ we find $R = L_{SC} + L_2 + L_3 = M_B$. For $\alpha \to -\infty$, $L_i^{-1/\alpha} \to 1$, hence $R \approx 2^{|\alpha|}$. The number 2 is indeed the largest number of bonds one must cut at each stage of iteration in order to break the structure into two pieces. Thus $z = \ln 2/\ln L$ for this fractal, where $L = L_1 + L_2 + L_4$.

For the same model,

$$\langle R^k \rangle_c = \langle r^k \rangle_c [L_{SC} + (L_2 L_3^{2k} + L_3 L_2^{2k})/(L_2 + L_3)^{2k}] \tag{6}$$

and the square bracket differs from (5) unless $L_2 = L_3$, or $k = (\alpha + 1)/2 = 0, 1$ or ∞. The difference $(\tilde{\zeta}(\alpha) - x[(\alpha + 1)/2])$ is maximal at $\alpha = 0$ and for $\alpha \to -\infty$.

The situation for $\alpha < 0$ is illustrated in figure 1. If current enters at the 'source' O (whose voltage is fixed to be V) and exits at the 'sink' G (whose voltage is fixed to be zero), then the ways to assign directions for currents through the bonds are restricted: the source (sink) has only outgoing (incoming) arrows; all other sites have at least one incoming and one outgoing arrow and there are no directed loops of arrows. Each assignment of arrows corresponds in the phase space of node voltages to a subregion defined by voltage inequalities, i.e. by current directions. On the boundary of such a subregion the voltage drop, ΔV, across some bond is zero. Since for $\alpha < 0$ the gradient with respect to node voltages of the quantity $(\alpha + 1)P$ is of order $|\Delta V|^\alpha \to +\infty$ at the boundary, one sees that within each subregion, i.e. for each arrow assignment, $(\alpha + 1)P$ attains a local minimum value corresponding to a solution to Kirchhoff's equations.

Consider now the limit $\alpha \to 0$. Starting at site O in figure 1, the incoming current splits, $I = I_{OG} + I_{OA}$, with $I_{OG}/I_{OA} = (V/\Delta V_{OA})^{1/\alpha}$. Since $V > V_{OA}$, $I_{OG}/I_{OA} \to \infty$ for $\alpha \to 0^+$ and $I_{OG}/I_{OA} \to 0$ for $\alpha \to 0^-$. Thus the whole current goes through OG, or the *minimal path*, for $\alpha \to 0^+$. In contrast, the current via OG is negligible for $\alpha \to 0^-$, and the whole current I reaches site A. Repeating this procedure at A we find that the full current goes through either AH or AB. Unless there are bonds with equal voltage drops, the full current goes through a SAW from O to G. On a walk with n bonds, $\Delta V_i = V/n$. Since the current chooses the smallest ΔV_i, it chooses the longest possible walk consistent with a given arrow assignment. Demanding that $P = (\Sigma_i r_i)|I|$ be a maximum now chooses that arrow assignment corresponding to the longest SAW (i.e. $n = 7$ in figure 1). This result may seem to apply only if all the ΔV_i at each vertex are different from each other. However, more detailed analysis shows that even when there are equivalent parallel paths (e.g. $L_2 = L_3$ in figure 2), the net resistance remains equal to the length of a longest SAW from the source to the sink.

The proof that $\tilde{\zeta}(-1) = D_B$ follows immediately from (2): for $\alpha \to -1$, R is given as $\Sigma_i r_i$, which is the total number of current-carrying bonds.

We now turn to the limit $\alpha \to -\infty$. From (2) $R = \Sigma_i r_i |i_i|^{\alpha+1}$ will be dominated by bonds which carry the smallest current $I_i = Ii_{\min}$. To locate these bonds we define 'cutting surfaces' for a cluster with a fixed source, sink and arrow assignment. Consider a domain wall dividing the cluster into two domains, one connected to the source, the other to the sink, such that all bonds cut by the domain wall carry current from the domain of the source to that of the sink. A cutting surface is a (possibly non-unique, as in figure 1(c)) domain wall which, for a given choice of source, sink and arrow assignments, cuts the maximum number $N_{\max}$ of bonds. Let $I_1, I_2 \ldots$ denote currents in the bonds cut by this surface for a solution to Kirchhoff's equations. For $\alpha \to -\infty$, the voltage drops occur essentially only over bonds that carry the minimal current. Then, by considering the voltage drops around a loop involving two different I_i one can argue that $\lim_{\alpha \to \infty} I_i/I_j = 1$. Thus the total current is $I = N_{\max} I_i$ so that $i_i = 1/N_{\max}$ and (2) yields $\lim_{\alpha \to -\infty} \ln R/|\alpha| = \ln N_{\max}$. We numerically verified this result for parts (a), (b) and (c) of figure 1 for which $N_{\max}$ is respectively 5, 5 and 4. Since we choose the arrow assignment with maximal P, the resistance corresponds to the maximal $N_{\max}$ as stated in conclusion (e) above.

To prove our statements on $\mathrm{d}\tilde{\zeta}/\mathrm{d}\alpha$, we start from (2):

$$\frac{\mathrm{d}R}{\mathrm{d}\alpha} = \sum_i \left[\ln |i_i| + (\alpha+1)\left(\frac{\partial \ln |i_i|}{\partial \alpha}\right)\right] r_i |i_i|^{\alpha+1}. \tag{7}$$

We now generalise Cohn's (1950) theorem to the non-linear case. Let $\varepsilon_{jk} = 1(-1)$ if the arrow on bond k leaves (enters) site j, and $\varepsilon_{jk} = 0$ otherwise. If the voltage at site j is $V(j)$, then $\Delta V_k = \Sigma_j \varepsilon_{jk} V(j)$ and $\Sigma_k h_k \Delta V_k = \Sigma_{kj} h_k \varepsilon_{jk} V(j)$. This sum vanishes if the h_k satisfy condition (a): $\Sigma_k h_k \varepsilon_{jk} = 0$. In our problem, $S_j \equiv \Sigma_k \varepsilon_{jk} |i_k| = 1(-1)$ if j is the source (sink), and $S_j = 0$ otherwise. Since $\Sigma_k \varepsilon_{jk}(\partial |i_k|/\partial \alpha) = \partial S_j/\partial \alpha = 0$ is condition (a) with $h_k = \partial |i_k|/\partial \alpha$,

$$\sum_k r_k |i_k|^{\alpha+1}\left(\frac{\partial \ln |i_k|}{\partial \alpha}\right) = \sum_k \left(\frac{\partial |i_k|}{\partial \alpha}\right) \Delta V_k = 0. \tag{8}$$

Thus (7) becomes $\mathrm{d}R/\mathrm{d}\alpha = \Sigma_i r_i |i_i|^{\alpha+1} \ln |i_i|$, or

$$\mathrm{d}\tilde{\zeta}(\alpha)/\mathrm{d}\alpha = \sum_i r_i |i_i|^{\alpha+1} \ln |i_i| / (R \ln L). \tag{9}$$

This derivative is therefore proportional to a weighted average of $\ln|i_i|$. Since $|i_i| = |I_i/I| \leq 1$, $\ln|i_i| \leq 0$ and $d\tilde{\zeta}/d\alpha \leq 0$. Equality holds only if terms in (9) with $|i_i| < 1$ are negligible. This is certainly the case for $\alpha \to \infty$, when the resistance is dominated by singly connected bonds (Blumenfeld and Aharony 1985), for which $i_i = 1$. This will also occur for $\alpha \to 0^{\pm}$, provided the symmetric sections (in which current splits into two parallel routes) exist only over a limited range of length scales. In the example of figure 1(a), the routes AHCDEFG and ABCDEFG are equvalent, and therefore the current will split between AHC and ABC, contributing $(\ln 2)/2$ to $-dR/d\alpha$ at $\alpha \to 0^-$. If $\tilde{\zeta}'(\alpha)$ at $\alpha = 0$ turns out to be non-zero, we would conclude that the average cluster is symmetric. Our series results (MBAH) show that this is not the case. The proof of (3) uses (8) and is very similar (MBAH).

We chose the approximant in (4) to reflect all the above. In a sense, it generalises (5). The parameters $(b-1)$, a and c measure the asymmetry, the fractal dimensionality and the relative size of the 'blobs', respectively. We note that the ε-expansion result (Harris 1986) has the same functional behaviour (exponential decay for $\alpha \to +\infty$, discontinuity at $\alpha = 0$) as our approximant.

To conclude: the study of the functional dependence of $\tilde{\zeta}$ on α and the resulting accurate approximant open new directions in studies of infinite sets of exponents and our new results for $\alpha < 0$ are useful in studying SAW on fractals.

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