Series analysis of randomly diluted nonlinear resistor networks

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The behavior of a randomly diluted network of nonlinear resistors, for each of which the voltage-current relationship is $|V|=r|I|^{\alpha}$, is studied with use of series expansions in the concentration p of conducting bonds on d-dimensional hypercubic lattices. The average nonlinear resistance $\langle R\rangle$ between pairs of sites separated by the percolation correlation length, scales as $|p-p_c|^{-\zeta(\alpha)}$. The exponent $\zeta(\alpha)$ was evaluated for $0<\alpha<\infty$ and $d=2$, 3, 4, 5, and 6, found to decrease monotonically from the exponent describing the minimal length, at $\alpha=0$, via that of the linear resistance, at $\alpha=1$, to the exponent characterizing the singly connected bonds, $\xi(\infty)\equiv 1$. Our results agree with known results for $\alpha=0$ and $\alpha=1$, also with recent results for general α at $d=6-\epsilon$ dimensions. The second moment $\langle R^2\rangle$ was found to diverge as $\langle R\rangle^2$ (for all α and d), indicating a scaling form for the probability distribution of R.

I. INTRODUCTION

A few years ago, Kenkel and Straley[1,2] proposed a model of a network of nonlinear resistors, each of which obeys the generalized Ohm's law

$$V=r|I|^{\alpha}\mathrm{sgn}I\ , \tag{1.1}$$

where V is the voltage drop across it and I is the current flowing through it, r is the nonlinear resistance, and α is the exponent characterizing the nonlinearity.

Consider now the (nonlinear) resistance $R_{\alpha}(L)$ between two terminals, a distance L apart, on the same cluster. Two of us recently showed[3] that this resistance reduces to the minimal ("chemical") path between these points, $L_{\min}$, for $\alpha\to 0$, and to the number of singly connected ("red") bonds between them, L_{SC}, for $\alpha\to\infty$. It trivially becomes the linear Ohmic resistance at $\alpha=1$.

In this paper we study the randomly diluted network on d-dimensional hypercubic lattices, where each bond (with resistance r) is randomly present with probability p or absent (with probability $1-p$). As p approaches the percolation threshold p_c, the percolation correlation length ξ diverges as $|p-p_c|^{-\nu}$, and all connected clusters are self-similar on length scales $L<\xi$. For such distances one expects the average resistance $\langle R_{\alpha}(L)\rangle$ to behave as a power of L:

$$\langle R_{\alpha}(L)\rangle\sim L^{\tilde{\zeta}(\alpha)}\ . \tag{1.2}$$

As L approaches ξ, this crosses over to

$$\langle R_{\alpha}(\xi)\rangle\sim\xi^{\tilde{\zeta}(\alpha)}\sim|p-p_c|^{-\zeta(\alpha)}\ , \tag{1.3}$$

with $\zeta(\alpha)=\tilde{\zeta}(\alpha)\nu$. On larger length scales there is practically no conductance for $p>p_c$, while the link-node-blob picture[4-6] implies that the conductivity scales as $\sigma\sim(p-p_c)^{\mu(\alpha)}$, with[1]

$$\mu(\alpha)=(d-1)\nu+[\zeta(\alpha)-\nu]/\alpha\ . \tag{1.4}$$

There exist many separate studies, using various techniques, which estimated $\zeta(\alpha)$ for $\alpha=0$ (Refs. 6–14) and $\alpha=1$.[15-24] Of particular interest is Coniglio's exact proof,[25] showing that $\zeta(\infty)=1$. Table I lists some of these previous estimates and compares them with our own estimates and at these values. It is easy to convince oneself that one always has $\zeta(0)\geq\zeta(1)\geq\zeta(\infty)$; the resistance of the minimal path decreases when more bonds are added in the blobs, and the resistance of the singly connected bonds increases when the blobs are added. Indeed, previous studies on various fractals showed[3] that $\zeta(\alpha)$ is always a monotonically decreasing function of α. This was also found in very recent ϵ expansions for $\zeta(\alpha)$, in $d=6-\epsilon$ dimensions.[26]

The aim of this paper is to use series expansions in p to obtain estimates of $\zeta(\alpha)$ for all d and α. This program is a systematic generalization of the earlier series studies for $\alpha=1$.[20-22] Up to this point we have discussed only the *average*, $\langle R_{\alpha}(L)\rangle$ or $\langle R_{\alpha}(\xi)\rangle$. In real experiments one should also worry about the fluctuations about this average, i.e., the *distribution* of resistances measured for pairs of terminals at fixed distance L (or ξ). To address this question, we also constructed series for the second moment $\langle R^2\rangle$.

This paper is organized as follows. Section II describes

TABLE I. Values of $\zeta(\alpha)$ for $\alpha=0,1$.

d	$\alpha=0$	$\alpha=1$	d	$\alpha=0$	$\alpha=1$
2	1.49±0.02[a]	1.297±0.007[l]	4	1.11±0.07[p]	1.05±0.02[r]
	1.52±0.01[b]	1.31±0.01[m]		1.03±0.08[c]	1.03±0.09[o]
	1.38±0.1[c]	1.29±0.01[n]		1.083±0.001[g]	1.053[q]
	1.54±0.05[d]	1.31±0.1[o]		1.10±0.04[d]	1.08±0.02[j]
	1.46±0.07[e]	1.26±0.02[p]		1.10±0.02[j]	1.06±0.01[k]
	1.509±0.004[f]	1.20±0.005[q]		1.11±0.01[k]	
	1.19±0.01[g]	1.32±0.08[j]			
	1.361±0.007[h]	1.31±0.05[k]			
	1.49[i]				
	1.50±0.08[j]		5	1.02±0.1[c]	1.02±0.02[r]
	1.48±0.05[k]			1.038[g]	1.01±0.08[o]
				1.04±0.01[j]	1.025[q]
3	1.19±0.04[b]	1.12±0.02[r]		1.04±0.01[k]	1.03±0.01[j]
	1.18±0.07[c]	1.16±0.07[o]			1.03±0.01[k]
	1.20±0.03[d]	1.084±0.001[q]			
	1.21[i]	1.06±0.1[s]			
	1.134±0.004[g]	1.32[t]	6	0.9±0.1[b]	1.01±0.02[r]
	1.12±0.05[h]	1.15±0.03[j]		1.02±0.02[c]	1.00±0.01[j]
	1.20±0.03[j]	1.12±0.03[k]		1.00±0.01[j]	1.00±0.01[k]
	1.19±0.03[k]			1.00±0.01[k]	

[a]Pike and Stanley, Ref. 6. Monte Carlo simulation.
[b]Alexandrowicz, Ref. 7. Monte Carlo simulation.
[c]Hong and Stanley, Ref. 8. Series expansion.
[d]Havlin and Nossal, Ref. 9. Monte Carlo simulation.
[e]Herrmann *et al.*, Ref. 10. Monte Carlo simulation.
[f]Grassberger, Ref. 11. Monte Carlo simulation.
[g]Janssen and Cardy, Grassberger, Ref. 13. ϵ expansion. The error bars for these ϵ-expansion results were obtained using different Padé estimates. Errors less than 0.001 in the ϵ expansions were ignored.
[h]Edwards and Kerstein, Ref. 14. Monte Carlo simulation.
[i]Ray, Ref. 12. Real-space renormalization.
[j]This work (ζ_1).
[k]This work ($\zeta_2/2$).
[l]Zabolitzky, Ref. 15; Lobb and Frank, Ref. 16. Transfer matrix.
[m]Herrmann *et al.*, Ref. 17. Transfer matrix.
[n]Hong *et al.*, Ref. 18. Monte Carlo simulation.
[o]Adler, Ref. 20. Series expansion.
[p]Essam and Bhatti, Ref. 21. Series expansion.
[q]Harris *et al.*, Lubensky and Wang, Ref. 19. ϵ expansion. The error bars for these ϵ-expansion results were obtained using different Padé estimates. Errors less than 0.001 in the ϵ expansions were ignored.
[r]Fisch and Harris, Ref. 22. Series expansion.
[s]Derrida *et al.*, Ref. 23. Transfer matrix.
[t]Mitescu and Greene, Ref. 24. Monte Carlo.

the construction of the series, and Sec. III presents the results of their analysis, which are summarized in Fig. 1 with specific results listed in Table III. In Sec. IV we analyze and discuss our results for $\langle R^2 \rangle$. Our conclusions are summarized in Sec. V.

II. SERIES CONSTRUCTION

In this section we follow closely Ref. 20. The percolation susceptibility is defined by

$$\chi_p = \left[\sum_j v_{ij} \right]_{\rm av} , \tag{2.1}$$

where v_{ij} is 1 if the two sites i and j belong to the same cluster and zero otherwise, and $[\ ]_{\rm av}$ denotes an average over all configurations of occupied and unoccupied bonds. The resistive susceptibility is defined by

$$\chi_R(\alpha) = \left[\sum_j R_{ij}(\alpha) v_{ij} \right]_{\rm av} , \tag{2.2}$$

where $R_{ij}(\alpha)$ is the (nonlinear) resistance between sites i and j.

We define $\chi_R(\alpha;\Gamma)$ for a cluster of sites, Γ, via

$$\chi_R(\alpha;\Gamma) = \sum_{i\in\Gamma} \sum_{j\in\Gamma} R_{ij}(\alpha) \tag{2.3}$$

in terms of which $\chi_R(\alpha)$ is obtained by summing over all clusters, weighting each cluster by its probability of occurrence. This is best done in terms of cumulants, whereby we may write

$$\chi_R(\alpha) = \sum_\Gamma W(\Gamma;d) p^{n_b(\Gamma)} \chi_R^c(\alpha;\Gamma) , \tag{2.4}$$

where $n_b(\Gamma)$ is the number of bonds in the diagram Γ, $W(\Gamma;d)$ is the number of ways per site a diagram topologically equivalent to Γ can be realized on a hypercubic lattice in d dimensions, and the sum is over all topologically inequivalent diagrams Γ. Also $\chi_R^c(\alpha;\Gamma)$ is the cumulant, defined by

$$\chi_R^c(\alpha;\Gamma) = \chi_R(\alpha;\Gamma) - \sum_{\gamma\in\Gamma} \chi_R^c(\alpha;\gamma) , \tag{2.5}$$

where the sum is over all subdiagrams, γ, of Γ.

The factor $p^{n_b(\Gamma)}$ in Eq. (2.4) implies that the evaluation of χ up to order p^n involves only clusters with up to n

TABLE II. The coefficients $A(k,l)$ for $\alpha=0.5$ and 5.0. Numbers enclosed in square brackets denote scale factors of powers of 10.

$\alpha=0.5$

$A(1,1)=0.100\,000\,000\,000\,00[+1]$	$A(2,1)=-0.200\,000\,000\,000\,00[+1]$	$A(2,2)=0.400\,000\,000\,000\,00[+1]$
$A(3,1)=0.300\,000\,000\,000\,00[+1]$	$A(3,2)=-0.120\,000\,000\,000\,00[+2]$	$A(3,3)=0.120\,000\,000\,000\,00[+2]$
$A(4,1)=0.126\,884\,198\,415\,26[+2]$	$A(4,2)=0.731\,158\,015\,847\,42[+1]$	$A(4,3)=-0.480\,000\,000\,000\,00[+2]$
$A(4,4)=0.320\,000\,000\,000\,00[+2]$	$A(5,1)=-0.717\,536\,793\,661\,02[+2]$	$A(5,2)=0.123\,507\,358\,732\,21[+3]$
$A(5,3)=0.332\,463\,206\,338\,93[+2]$	$A(5,4)=-0.159\,999\,999\,999\,99[+3]$	$A(5,5)=0.799\,999\,999\,999\,96[+2]$
$A(6,1)=-0.274\,910\,165\,301\,55[+3]$	$A(6,2)=0.117\,860\,244\,799\,50[+3]$	$A(6,3)=0.291\,310\,958\,600\,36[+3]$
$A(6,4)=0.159\,738\,961\,901\,69[+3]$	$A(6,5)=-0.480\,000\,000\,000\,00[+3]$	$A(6,6)=0.192\,000\,000\,000\,00[+3]$
$A(7,1)=0.250\,389\,733\,781\,58[+4]$	$A(7,2)=-0.507\,730\,236\,069\,14[+4]$	$A(7,3)=0.212\,571\,269\,959\,98[+4]$
$A(7,4)=0.684\,721\,758\,204\,65[+3]$	$A(7,5)=0.665\,970\,565\,071\,26[+3]$	$A(7,6)=-0.134\,400\,000\,000\,00[+4]$
$A(7,7)=0.448\,000\,000\,000\,01[+3]$	$A(8,1)=0.191\,518\,545\,718\,07[+5]$	$A(8,2)=-0.262\,959\,627\,957\,87[+5]$
$A(8,3)=0.252\,849\,850\,257\,65[+4]$	$A(8,4)=0.322\,202\,539\,284\,38[+4]$	$A(8,5)=0.152\,065\,791\,588\,22[+4]$
$A(8,6)=0.244\,092\,641\,267\,79[+4]$	$A(8,7)=-0.358\,400\,000\,000\,00[+4]$	$A(8,8)=0.102\,400\,000\,000\,00[+4]$
$A(9,1)=-0.158\,601\,998\,719\,93[+6]$	$A(9,2)=0.392\,704\,713\,022\,59[+6]$	$A(9,3)=-0.311\,790\,709\,499\,93[+6]$
$A(9,4)=0.701\,750\,095\,384\,85[+5]$	$A(9,5)=0.336\,038\,820\,271\,34[+4]$	$A(9,6)=0.294\,977\,406\,563\,81[+4]$
$A(9,7)=0.812\,382\,339\,042\,75[+4]$	$A(9,8)=-0.921\,600\,000\,000\,02[+4]$	$A(9,9)=0.230\,400\,000\,000\,01[+4]$
$A(10,1)=-0.184\,379\,783\,331\,60[+7]$	$A(10,2)=0.342\,085\,513\,629\,45[+7]$	$A(10,3)=-0.161\,985\,445\,332\,98[+7]$
$A(10,4)=-0.105\,371\,458\,782\,25[+6]$	$A(10,5)=0.138\,535\,815\,808\,08[+6]$	$A(10,6)=-0.169\,731\,805\,435\,38[+4]$
$A(10,7)=0.409\,652\,346\,888\,39[+4]$	$A(10,8)=0.251\,635\,879\,109\,94[+5]$	$A(10,9)=-0.230\,399\,999\,999\,99[+5]$
$A(10,10)=0.511\,999\,999\,999\,98[+4]$	$A(11,1)=0.131\,165\,690\,600\,52[+8]$	$A(11,2)=-0.379\,672\,310\,393\,74[+8]$
$A(11,3)=0.395\,741\,204\,383\,10[+8]$	$A(11,4)=-0.169\,921\,937\,818\,34[+8]$	$A(11,5)=0.198\,815\,199\,720\,98[+7]$
$A(11,6)=0.277\,502\,604\,119\,33[+6]$	$A(11,7)=-0.248\,144\,519\,178\,89[+5]$	$A(11,8)=-0.828\,884\,647\,254\,37[+3]$
$A(11,9)=0.737\,910\,580\,822\,75[+5]$	$A(11,10)=-0.563\,200\,000\,000\,03[+5]$	$A(11,11)=0.112\,640\,000\,000\,01[+5]$

$\alpha=5.0$

$A(1,1)=0.100\,000\,000\,000\,00[+1]$	$A(2,1)=-0.200\,000\,000\,000\,00[+1]$	$A(2,2)=0.400\,000\,000\,000\,00[+1]$
$A(3,1)=0.300\,000\,000\,000\,00[+1]$	$A(3,2)=-0.120\,000\,000\,000\,00[+2]$	$A(3,3)=0.120\,000\,000\,000\,00[+2]$
$A(4,1)=0.158\,324\,580\,830\,73[+2]$	$A(4,2)=0.416\,754\,191\,692\,68[+1]$	$A(4,3)=-0.480\,000\,000\,000\,00[+2]$
$A(4,4)=0.320\,000\,000\,000\,00[+2]$	$A(5,1)=-0.843\,298\,323\,322\,90[+2]$	$A(5,2)=0.148\,659\,664\,664\,58[+3]$
$A(5,3)=0.206\,701\,676\,677\,05[+2]$	$A(5,4)=-0.159\,999\,999\,999\,99[+3]$	$A(5,5)=0.799\,999\,999\,999\,96[+2]$
$A(6,1)=-0.344\,147\,549\,312\,37[+3]$	$A(6,2)=0.182\,302\,827\,021\,35[+3]$	$A(6,3)=0.333\,834\,219\,287\,91[+3]$
$A(6,4)=0.122\,010\,503\,003\,12[+3]$	$A(6,5)=-0.480\,000\,000\,000\,00[+3]$	$A(6,6)=0.192\,000\,000\,000\,00[+3]$
$A(7,1)=0.306\,266\,732\,258\,91[+4]$	$A(7,2)=-0.622\,844\,475\,811\,10[+4]$	$A(7,3)=0.272\,405\,821\,102\,25[+4]$
$A(7,4)=0.779\,357\,883\,157\,74[+3]$	$A(7,5)=0.565\,361\,341\,341\,72[+3]$	$A(7,6)=-0.134\,400\,000\,000\,000\,00[+4]$
$A(7,7)=0.448\,000\,000\,000\,01[+3]$	$A(8,1)=0.237\,363\,906\,572\,32[+5]$	$A(8,2)=-0.330\,786\,208\,767\,49[+5]$
$A(8,3)=0.390\,871\,440\,345\,12[+4]$	$A(8,4)=0.403\,269\,847\,790\,26[+4]$	$A(8,5)=0.177\,941\,398\,480\,88[+4]$
$A(8,6)=0.218\,940\,335\,335\,42[+4]$	$A(8,7)=-0.358\,400\,000\,000\,00[+4]$	$A(8,8)=0.102\,400\,000\,000\,00[+4]$
$A(9,1)=-0.196\,875\,662\,152\,44[+6]$	$A(9,2)=0.485\,193\,204\,857\,08[+6]$	$A(9,3)=-0.385\,059\,840\,765\,06[+6]$
$A(9,4)=0.881\,391\,653\,345\,87[+5]$	$A(9,5)=0.429\,710\,161\,252\,04[+4]$	$A(9,6)=0.370\,686\,306\,526\,28[+4]$
$A(9,7)=0.752\,016\,804\,805\,03[+4]$	$A(9,8)=-0.921\,600\,000\,000\,02[+4]$	$A(9,9)=0.230\,400\,000\,000\,01[+4]$
$A(10,1)=-0.229\,269\,583\,236\,02[+7]$	$A(10,2)=0.427\,223\,680\,789\,27[+7]$	$A(10,3)=-0.206\,284\,351\,160\,12[+7]$
$A(10,4)=-0.936\,301\,959\,120\,18[+5]$	$A(10,5)=0.165\,985\,863\,922\,39[+6]$	$A(10,6)=-0.116\,926\,435\,957\,38[+4]$
$A(10,7)=0.629\,107\,363\,913\,53[+4]$	$A(10,8)=0.237\,550\,587\,787\,81[+5]$	$A(10,9)=-0.230\,399\,999\,999\,99[+5]$
$A(10,10)=0.511\,999\,999\,999\,98[+4]$	$A(11,1)=0.163\,754\,837\,238\,00[+8]$	$A(11,2)=-0.471\,715\,784\,307\,61[+8]$
$A(11,3)=0.490\,241\,472\,587\,84[+8]$	$A(11,4)=-0.211\,083\,349\,336\,38[+8]$	$A(11,5)=0.255\,796\,342\,480\,59[+7]$
$A(11,6)=0.318\,194\,130\,816\,42[+6]$	$A(11,7)=-0.267\,031\,336\,602\,39[+5]$	$A(11,8)=0.532\,339\,693\,011\,20[+4]$
$A(11,9)=0.705\,715\,629\,229\,37[+5]$	$A(11,10)=-0.563\,200\,000\,000\,03[+5]$	$A(11,11)=0.112\,640\,000\,000\,01[+5]$

bonds. We constructed the series for (2.1) and (2.2) up to 11th order in p. While computing the percolation susceptibility for each cluster is trivial, the resistive susceptibility involves the nontrivial task of solving the nonlinear Kirchoff's equations based on Eq. (1.1). For reasons of computational efficiency we solved the set of Kirchoff's equations differently for $\alpha>1$ and for $\alpha<1$. For $\alpha>1$ we solved the equations for the potentials at the sites (node analysis[27]), while for $\alpha<1$ we solved for the currents through the bonds (loop analysis[27]). In both methods the equations were solved iteratively, where as an initial guess we used the solution for the neighboring α. In the vicinity of $\alpha=1$, where both methods were used, we find the same results.

Writing

$$\chi_R(\alpha)=\sum_{k,l}A(k,l)d^l p^k\,, \tag{2.6}$$

the coefficients $A(k,l)$ for $\alpha=0.5$ and 5.0 are shown in Table II. For $\alpha=1$ the coefficients are the same as in Refs. 20 and 22, while for $\alpha=0$ and $\alpha\to\infty$ (the coefficients seem to be essentially constant for $\alpha>10$), they are the same as the coefficients for the minimal chemical length and the length of the singly connected bonds in

Ref. 8.

The average resistance $\langle R_\alpha \rangle$ is defined by

$$\langle R_\alpha \rangle = \frac{\chi_R}{\chi_p} \sim |p - p_c|^{-\zeta(\alpha)} . \tag{2.7}$$

In order to find $\zeta(\alpha)$ we analyzed the series for both χ_R and χ_Γ, and $\zeta(\alpha)$ was found as the difference of the two corresponding exponents, $\zeta = \gamma_R - \gamma$. The series for the ratios χ_R/χ_p were less well behaved than those for χ_R or χ_p.

III. ANALYSIS AND DISCUSSION

We analyzed the series using the "nonhomogeneous differential Padé method."[28] By this method we obtain 20–60 estimates for p_c and the critical exponent γ. The estimates for γ and γ_R were plotted as a function of the estimates for p_c. The resulting smooth functions $\gamma(p_c)$ and $\gamma_R(p_c)$ were fitted by linear curves. By interpolating to the known value of p_c (see Table III) we obtained estimates for γ and γ_R. The error bars in the exponents were taken as those of the interpolation procedure, where we have taken into account the uncertainty in the values of p_c. Explicit values of $\zeta(\alpha)$ for $\alpha = 0.5$ and 5.0 in 2–6 dimensions are shown in Table III, along with the error bars. The errors seem to decrease with increasing dimensionality. For $d > 6$ the blobs become irrelevant, and $\zeta(\alpha) = 1$ for all α.[19,26,31]

The results in the literature for $\alpha = 0$ and 1 (discussed previously) are shown in Table I, where they are also compared with the estimates found by our analysis. The agreement suggests that our estimates for other values of α are also quite reliable.[32] Figure 1 shows $\zeta(\alpha)$ as a function of $\alpha/(\alpha+1)$ for $d = 2,3$. We see that $\zeta(\alpha)$ decreases monotonically and continuously from $\zeta(0)$, which corresponds to the minimal-chemical length, through $\zeta(1)$, which corresponds to the linear resistance, to the exact $\zeta(\infty) = 1$, which corresponds to the length of the singly connected bonds.

In Table III we also compare our calculations at $d = 5$ with the recent ϵ-expansion results.[26] For large α, where $\zeta(\alpha)$ is close to unity, our results do not provide a very sensitive test. For $\alpha < 1$ the situation is more favorable and at $\alpha = \frac{1}{2}$ our results agree with the ϵ-expansion results. Longer series will probably be able to provide a definitive numerical test of the ϵ expansion. We also determined $d\zeta(\alpha)/d\alpha$ for $\alpha = 1$ and $d = 5$, from a plot similar to those in Fig. 1. Our result $d\zeta(\alpha)/d\alpha = -0.012$ is quite close to the value, $-\epsilon/72$, for this quantity in Ref. 26. It would be desirable to test the predictions for $d\zeta(\alpha)/d\alpha$ for α not near unity in order to distinguish between the predictions of Refs. 26 and 33, which differ only for $\alpha \neq 1$.

TABLE III. Values of p_c and results for $\alpha = 0.5$ and 5.0.

		$\alpha = 0.5$		$\alpha = 5.0$	
d	p_c	ζ	$\zeta_2/2$	ζ	$\zeta_2/2$
2	$\frac{1}{2}$ (exact)	1.41±0.08	1.40±0.05	1.02±0.08	1.02±0.02
3	0.2486[a]	1.18±0.04	1.15±0.04	1.02±0.02	1.01±0.02
4	0.1601[b,c]	1.09±0.02	1.11±0.02	1.02±0.02	1.01±0.01
5	0.1181[b,c]	1.04±0.01	1.04±0.01	1.02±0.01	1.01±0.01
		1.035[d]	1.035[d]	1.001[d]	1.00[d]
6	0.0941[c]	1.01±0.01	1.02±0.02	1.00±0.01	1.00±0.01

[a]Grassberger, Ref. 29.
[b]Fisch and Harris, Ref. 20.
[c]Adler *et al.*, Ref. 30.
[d]ϵ expansion, Harris, Ref. 26.

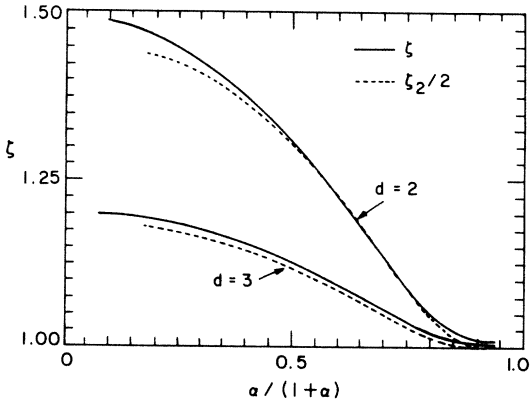

FIG. 1. Series results for $\zeta(\alpha)$ and $\zeta(\alpha)/2$.

IV. SECOND MOMENT AND SECOND CUMULANT

We now consider the behavior of the second moment of the resistance, $\langle R^2 \rangle$. The corresponding susceptibility is defined by

$$\chi_{R^2} = \left[\sum_j R_{ij}^2 v_{ij} \right]_{\rm av} , \tag{4.1}$$

and we expect it to diverge at p_c with exponent ζ_2. By the same method as above we constructed and analyzed series for χ_{R^2}. Figure 1 also shows our results for $\zeta_2/2$ versus $\alpha/(\alpha+1)$ for $d = 2,3$ and Table III contains explicit values of ζ_2 for $\alpha = 0.5$ and 5.0 in 2–6 dimensions. We see that within our accuracy, the exponents ζ and ζ_2 obey the relation

$$\zeta_2(\alpha) = 2\zeta(\alpha) . \tag{4.2}$$

We also studied the second cumulant, defined by

$$\langle R^2 \rangle_c = \langle R^2 \rangle - \langle R \rangle^2 . \tag{4.3}$$

A series for the second-cumulant susceptibility can be obtained by the susceptibilities discussed above, by

$$\chi^c_{R^2} = \chi_{R^2} - \frac{(\chi_R)^2}{\chi_p} . \tag{4.4}$$

(An interesting quantity to study is $\sum_j \chi^{(2)}_{ij}$ where $\chi^{(2)}_{ij} = [v_{ij}(R_{ij} - \langle R_{ij} \rangle)^2]_{\rm av}$ where $\langle R_{ij} \rangle = [v_{ij} R_{ij}]_{\rm av}/[v_{ij}]_{\rm av}$. However, this quantity is not one for which one can easily develop a series expansion.) This series is less well behaved than the other series we studied and the error bars for ζ^c_2 are quite large. However, within the limits of our accuracy, we can say that the relation

$$\zeta^c_2(\alpha) = 2\zeta(\alpha) \tag{4.5}$$

is consistent with our results.

Let $P(R,\mathbf{x},p)$ denote the conditional probability that the (nonlinear) resistance between two sites separated by a displacement $\mathbf{x}$ assumes the value R, subject to the two sites being in the same cluster. Then the relation $\zeta_k = k\zeta$, implies that P has the scaling form

$$P(R,\mathbf{x},p) = f(R/rx^{\tilde{\zeta}}, x/\xi)/rx^{\tilde{\zeta}} , \tag{4.6}$$

where r is the resistance of a single resistor. For the linear case, this form was found numerically,[34] and is implied by the scaling behavior of the resistive correlation functions in the field theories.[19] For the nonlinear case considered here this relation is probably also implied by the field-theoretic formulation.[26] The present work provides the first numerical evidence in support of Eq. (4.6) for $\alpha \neq 1$.

Equation (4.2) suggests that $\zeta_2/2$ should also serve as an estimate for ζ. These estimates appear in Tables I and III. Series estimates for ζ_k/k for different k's, which give much better estimates for ζ, will appear elsewhere.

V. CONCLUSION

In this paper we studied the nonlinear resistance problem.[1-3] We confirmed by the series-expansion method that this problem interpolates smoothly between two known topological problems: the minimal-chemical length (for $\alpha = 0$) and the singly connected bonds (for $\alpha \to \infty$). We studied the second moment and the second cumulant of the resistance, and concluded that the critical exponents corresponding to these two are the same and equal to twice the critical exponent of the resistance, $\tilde{\zeta}(\alpha)$. This equality suggests that even for $\alpha \neq 1$, the probability function for the resistance between two points has the scaling form written in Eq. (4.6).

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